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MODELS IN THE THEORY OF SEVERAL COMPLEX VARIABLES

STEFAN BERGMAN, Harvard University*

1. Introduction. The theory of functions of a single complex variable has exerted a valuable unifying and systematizing influence on the field of analysis, even in domains which are primarily concerned with real function theory. This is especially true in such fields as analytic number theory, the theory of linear differential equations, power series, algebraic functions, *etc.*, *etc.*; more important still it has been a powerful instrument of research in both pure and applied nathematics, in mathematical physics, and in engineering.

Many leading mathematicians—Weierstrass, Poincaré, Picard, and Osgood being among the most eminent—appear to have felt that a theory of analytic functions of several complex variables would be similarly valuable in various other branches of mathematics, say, non-linear differential equations, and that such a theory might well be the key to many important but as yet unsolved problems. Nevertheless, while the theory of analytic functions of one complex variable has been making a continuous and rapid progress and has become one of the most important and useful tools in the standard equipment of the mathematician, the theory of functions of several complex variables has by no means enjoyed such spectacular development; its progress is slow, halting and unintegrated; and relatively few mathematicians have any considerable knowledge of this latter field.

There are many reasons why there should be such a marked disparity in the states of advancement of these two theories, despite their close affinities. Perhaps the most obvious is the greater formal complexity of the functions of many variables; in the author's opinion one of the most significant factors is to be found in our lack of direct intuition of spaces of more than three dimensions.

In the first half of the nineteenth century the theory of analytic functions of a complex variable was formal, manipulative and very difficult. The modern reader is frequently baffled by the complications of analysis and the demands made on the ingenuity of the mathematician of that time to prove theorems which do not seem difficult to the contemporary mathematician. Much of the credit for this simplification and increased power must be ascribed to the introduction and elaboration of geometric methods in function theory. The mathematician's innate geometric intuition serves him by enabling him to interpret many involved and frequently subtle analytic properties of functions as relations between easily grasped geometric configurations, while the use of geometric language renders his statements both clearer and briefer.

This invaluable aid is almost totally wanting when we attack problems of functions of many variables. This note discusses a possible means of circumventing some of the difficulties of picturing spaces of more than three dimensions.

* The author wishes to express his thanks to Mr. M. H. Slud for friendly criticism and many helpful suggestions.

It will be shown how models* may be used for the representation of four-dimensional manifolds encountered in the theory of functions of two complex variables, and how appropriate training can create a kind of intuition for domains in space of four and more dimensions.

2. The role of geometrical interpretation. Interpretation of complex numbers as points in the x, y-plane has proved highly fruitful in the theory of functions of a complex variable. It is only an easy step beyond this to interpret the domain of definition of a function as a region in the plane and to express theorems on analytic functions in geometric terms. For instance, one of the most important and fundamental results in the theory of functions may be put into the following simple geometric form: Every simply connected domain with at least two boundary points may be mapped bi-uniquely and conformally on a circle. Naturally these (geometric) theorems are correctly and completely expressive of the full (analytic) content only if the geometric notions have been axiomatized, *i.e.* described in a completely abstract manner. Despite this the geometric formulation possesses various advantages connected with the fact that we do have intuition of two-dimensional space.

As mentioned before, while operating with ideas we must be able to keep them constantly in mind; as we often have a better understanding of geometrical relationships than of the same relationships in abstract terms† it goes without saying that whenever one must work with something which allows of geometrical representation, a geometrically minded mathematician will prefer to use the geometrical model. Many qualitative properties then become apparent such as, for example, whether the domain is connected or consists of several pieces, whether a curve is continuous or discontinuous, whether the domain is convex or not, or what symmetry properties it possesses, *etc.* Most properties which are associated with first and second derivatives may be seen in this way, such as whether a curve has a continuously turning tangent or corners and cusps, and whether there are maxima and minima. If several configurations $A, B, C \cdots$ are given then we can readily see the nature and properties of the configurations symbolized by $A+B+C+\cdots$, $A \subset B \subset C$, $A \cap B$, *etc.*

Moreover if the elements have been ordered in some manner, we can immediately see their mutual relations. To a certain extent we can also apprehend certain metric relations, such as estimating which of two curves is longer or which of two domains has the larger area.

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* Models of this kind are described in the author's papers: 1. Zur Veranschaulichung der Kreiskörper und Bereiche mit ausgezeichneter Randfläche, Jahresbericht der deutschen Mathem. Vereinigung vol. 42, 1933, pp. 238–252; 2. The visualization of domains of the theory of functions of two complex variables, Journal of Mathematics and Physics (M.I.T.) vol. 20, 1941, pp. 107–117.

† An obvious analogy is the difference between hearing a remark in one's native language or hearing the same statement in a language in which one is not altogether at ease. The second case demands considerable more mental effort.

3. Models in abstract mathematical theory. In Section 2 we have indicated the advantages for a geometrically minded mathematician, of using appropriate models in the theory of functions of one variable (two-dimensional case). Before discussing models in four-dimensional space it would be useful to clarify the relationship between the abstract geometrical notion and a model.

In order to describe geometrical configurations verbally or analytically, we often need a very considerable number of words or symbols. The most convenient method of describing a geometrical manifold is frequently by making use of equalities and inequalities.

For example, the set of points on the straight line which bisects the angle of the first quadrant may be represented by the symbol $E[x-y=0]$;* and the interior of the unit circle with center at the origin may be denoted by $E[x^2+y^2<1]$. Unfortunately, it is rarely possible to characterize a manifold so simply. It would be difficult to give an analytic expression for the interior of a polyhedron, while it would be impossible to characterize many of the manifolds used by the analyst with a finite number of symbols. Even in those cases where an analytic expression can be found, it is usually very long, and it is all but impossible to deduce the geometric properties of the figure from its analytic representation.† These difficulties are often circumvented by simply using an arbitrary symbol of some sort, most usually a letter, to denote the configurations which occur most commonly in mathematical discussions. Of course the symbol used has none of the properties of the entity it represents, any more than the letters of a word in any way resemble the object to which the word refers.

A further step in this direction is the use of geometric models. (Two-dimensional geometric models are usually called diagrams.) Naturally we may regard a model as a sort of symbol for the abstract entity known as a geometric manifold, but it is much more than that. The model (diagram) is a material object which has certain of the properties of the geometric concept it represents, and that is why it is useful. The representation is never perfect, however. The model seldom has *all* the properties of the original, and at the same time it has properties of its own which its original does not possess. Therefore care must be taken that the properties of the model shall not be mistaken for the properties of the original, a very common source of error. Because almost all geometric concepts were originally suggested by the properties of real figures, it is customary to say that the abstract concept is an "idealization" of the model. After the

* By $E[\cdots]$ we denote a set of points whose coordinates satisfy the equalities or inequalities indicated in brackets.

† We wish to emphasize that the ability to deduce properties depends to a large extent upon the individual, and can be increased by appropriate training. The question of whether in the future some "machines" can be constructed which will facilitate our operating with long formulae, and automatically give us information about the properties of the manifold remains at the present time open.

abstract concept has become familiar, the model is regarded as a "realization" of the concept.

The most important disadvantage of diagrams is that they are not able to describe the manifold completely, *e.g.* to indicate whether a circle is to be taken with its boundary or without it. Because of this during the transition from the diagram to the abstract notion we often commit mistakes by attributing to the manifold properties which are possessed only by the model.

A means of checking our conclusions would be to repeat our reasoning but now operating exclusively with abstract symbols, *e.g.* of the type indicated earlier in this section. This situation necessitates a certain kind of training and particular flexibility of mind, enabling one to pass from an intuitive model to an abstract formulation of notions and back.

These difficulties arise naturally in operating with models of four-dimensional space; in addition to this there occur further difficulties, which we shall indicate in what follows.

4. The advantage of models in teaching and research. Despite the difficulties mentioned before it seems that in many cases the use of models for a geometrically minded individual is of great value in research. Two important reasons may perhaps be indicated:

First, our interest in a thing is determined to a considerable extent by the amount of mental energy which is required for working with it. We involuntarily prefer to think in terms of familiar things which can be conceived without special effort. In other words, the extent of our interest in any subject depends to a certain degree on our previous associations—and here again we see the advantage of being able to use intuitive geometry to help in reasoning about abstract relations.

Second, if a relationship is known, it is not too important how we formulate it, provided only that our formulation is sufficiently precise. Research, on the other hand, is the search for new relations and in that case it becomes highly significant in which way our problems are formulated. If we can succeed in stating our problem in terms of a class of objects with which we have gained great familiarity, then we may be able to exploit our previous experience with these things. The form of the problem in the new representation may suggest comparisons and hypotheses and further questions, and our previous experience will enable us to see new relationships all the more readily. Thus, the moment that complex numbers are regarded as points in a plane, the analogy with integrals of real functions leads to the formulation of curvilinear integrals in the complex plane—the most fruitful approach in the theory of functions. Furthermore, even after some fact has been discovered and it has been reduced to abstract form which completely and rigorously describes it, even then, the intuitive processes by means of which we have worked are of importance in further research because of the ready associations and the many problems which in this way arise almost spontaneously.

5. Models of four-dimensional manifolds. It would, perhaps, be of interest to illustrate the foregoing by a description of a type of model used for visualization of four-dimensional space.*

These models are moving three-dimensional pictures, that is to say time is chosen as one of the coordinates. A four-dimensional domain is then represented as a moving picture. The continuous process is thus recorded by a finite number of "frames," and this last is in turn often replaced by a number of stills.

In order to have a better idea as to how intuitive these models are, let us put ourselves in the position of beings ("Flatlanders") who are able to "see" two-dimensional domains but who do not possess intuition for three-dimensional space.

FIG. 1. A sphere in an isometric representation.

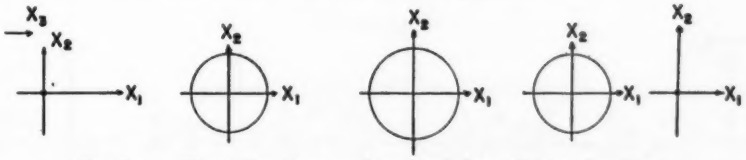

FIG. 2a–2e. Successive plane sections of a sphere. (Series of sections perpendicular to the x_1-axis.)

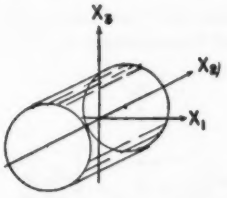

FIG. 3. A cylinder in an isometric representation.

Let us consider two examples of corresponding models of three-dimensional space, namely that of a cylinder $E[-1<x_2<1,\ x_1^2+x_3^2<1]$ and of a sphere $E[x_1^2+x_2^2+x_3^2<1]$.

Assuming that x_3 has been chosen as time we obtain films whose "stills" are represented in Figures 2a–2e and 4a–4e.

Obviously Figures 1 and 3 are for us much "better" representations of the above manifolds. However this impression is mostly due to the fact that we *do*

* Naturally various other types of models are possible and it depends on the purpose what type will be most suitable.

have intuition for three-dimensional space and are sufficiently trained to "imagine" from a two-dimensional projection (see Figures 1 and 3) the corresponding three-dimensional manifold. In the four-dimensional case, the corresponding projection of a four-dimensional manifold on the three-dimensional space (without the long training necessary to "reading" such projections) will give us almost no information about the properties of the four-dimensional manifold.

In Figures 5a–5e and 6a–6e stills of the models described above of the "hypersphere" $E[x_1^2+x_2^2+x_3^2+x_4^2<1]$ and of the "bicylinder" $E[x_1^2+x_2^2<1, x_3^2+x_4^2<1]$ are given.

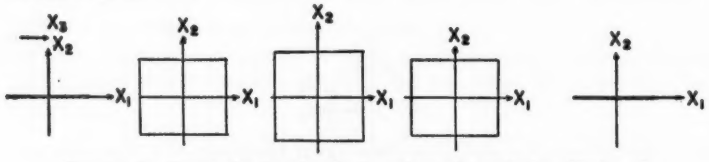

FIG. 4a–4e. Successive longitudinal sections of a cylinder. (Series of sections perpendicular to the x_1-axis.)

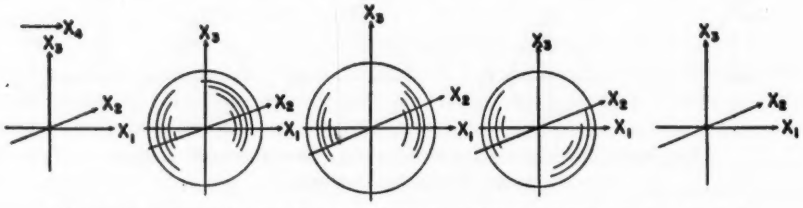

FIG. 5a–5e. Successive sections of a (four-dimensional) hypersphere by a series of (three-dimensional) hyperplanes perpendicular to the x_4-axis.

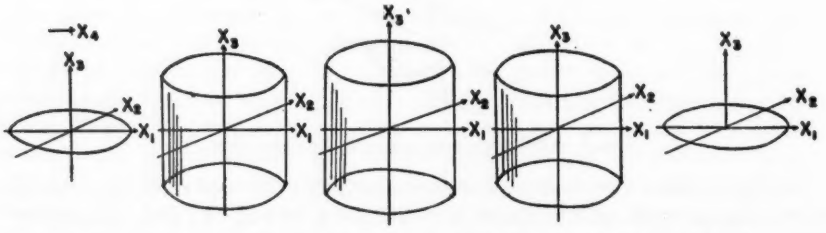

FIG. 6a–6e. Successive sections of a (four-dimensional) bicylinder by a series of (three-dimensional) hyperplanes perpendicular to the x_4-axis.

6. The training necessary to "read" models. As mentioned before the above (or other) models can be successfully used only after sufficiently long training in "reading" the properties of the manifold represented by the model.

Therefore the problem arises of working out a program for such training, in particular in connection with our models. Obviously this is a very complicated

question of a partially psychological nature, and cannot be answered without extensive experimental study.

It is the opinion of the author (who has only limited experience in this direction) that training should proceed in such a way that we systematically construct at first the stills for a series of various manifolds which are of interest in the theory of functions of two complex variables.*

The next stage of such training should consist of working out rules for changing a model if the manifold which it represents undergoes a linear transformation, at first say a translation or rotation, and later more complicated transformations.

Some preliminary investigations of this kind can be found in the papers mentioned in the footnote (page 496). On the other hand only actual use of a great variety of models can determine their value as well as give further indications concerning the lines along which training should proceed.

It seems therefore that this investigation should proceed on a considerably larger scale, and use more powerful means, *e.g.*, actual moving pictures.

The author hopes that the present article will contribute toward the realization of such a program.

AN INEQUALITY OF JENSEN

E. F. BECKENBACH, University of California, Los Angeles

1. Introduction. Let (a) denote a set $a_1, a_2, \cdots, a_n$ of n positive values, $n \geqq 2$. It has been shown by Pringsheim [7], who attributed his proof to Lüroth, and by Jensen [5, p. 192], that for $r > 1$ we have

$$\left(\sum_{j=1}^{n} a_j^r \right)^{1/r} < \sum_{j=1}^{n} a_j. \tag{1}$$

We write

$$\left(\sum_{j=1}^{n} a_j^r \right)^{1/r} \equiv \mathfrak{S}_r(a), \qquad (r \neq 0). \tag{2}$$

If in (1) we replace a_j by a_j^s, for $s > 0$, and rs by t, and take the sth root of both members of the inequality, we obtain

$$\mathfrak{S}_t(a) < \mathfrak{S}_s(a), \qquad (t > s > 0). \tag{3}$$

The inequality (3) is sometimes called [3; 4, p. 28] *Jensen's inequality*; it can be obtained directly as follows [4, p. 4]:

* We omit here the discussion of models of n-dimensional space for $n > 4$. We wish however to indicate that diagrams similar to $2a - 2e$ and $4a - 4e$ with several series of stills representing the intersections of the manifold with $x_4 = \text{const.}$, those with $x_3 = \text{const.}$, *etc.* can be constructed and used as models of manifolds of a space with more dimensions.

$$\frac{\left(\sum_{j=1}^{n} a_j^t\right)^{1/t}}{\left(\sum_{j=1}^{n} a_j^s\right)^{1/s}} = \left[\sum_{j=1}^{n} \frac{a_j^t}{\left(\sum_{k=1}^{n} a_k^s\right)^{t/s}}\right]^{1/t} = \left[\sum_{j=1}^{n} \left(\frac{a_j^s}{\sum_{k=1}^{n} a_k^s}\right)^{t/s}\right]^{1/t}$$

$$< \left(\sum_{j=1}^{n} \frac{a_j^s}{\sum_{k=1}^{n} a_k^s}\right)^{1/t} = 1.$$

Let

$$m \equiv \min_{j=1,2,\cdots,n} a_j, \qquad M \equiv \max_{j=1,2,\cdots,n} a_j.$$

From (2) we obtain

$$M < \mathfrak{S}_t(a) < n^{1/t}M, \qquad (t > 0),$$

so that [4, p. 29]

$$\lim_{t\to+\infty} \mathfrak{S}_t(a) = M.$$

Since

$$\lim_{t\to 0} \sum_{j=1}^{n} a_j^t = n,$$

it follows [4, p. 29] that

$$\lim_{t\to+0} \mathfrak{S}_t(a) = +\infty.$$

Hence $\mathfrak{S}_t(a)$ *decreases from* $+\infty$ *to* M *as* t *increases from* $+0$ *to* $+\infty$.

Since

$$\mathfrak{S}_{-t}(a) = \frac{1}{\mathfrak{S}_t(a)},$$

it follows from the above results that $\mathfrak{S}_t(a)$ *decreases from* m *to* 0 *as* t *increases from* $-\infty$ *to* -0.

On the other hand, the mean value function

$$\mathfrak{M}_t(a, q) \equiv \left(\sum_{j=1}^{n} q_j a_j^t\right)^{1/t}, \qquad (t \neq 0),$$

$$\mathfrak{M}_0(a, q) \equiv \prod_{j=1}^{n} a_j^{q_j},$$

where $q_1, q_2, \cdots, q_n$ are fixed positive values with

$$\sum_{j=1}^{n} q_j = 1,$$

increases [4, p. 26] *continuously from m to M, or remains constant if all the a's are equal, as t increases from* $-\infty$ *to* $+\infty$.

It is known [4, pp. 27, 72] that the function

$$\log \left[\mathfrak{M}_t(a, q)\right]^t \equiv \log \sum_{j=1}^{n} q_j a_j^t \tag{4}$$

is a convex function of t for $-\infty < t < +\infty$. Since we have

$$\log \sum_{j=1}^{n} a_j^t \equiv \log \left[\mathfrak{M}_t(a, 1/n)\right]^t + \log n, \tag{5}$$

it follows [4, p. 28] that the continuous function (5), which coincides with $\log \left[\mathfrak{S}_t(a)\right]^t$ for $t \neq 0$, *is a convex function for t for* $-\infty < t < +\infty$.

We offer the following alternative to the proofs usually given of the convexity of (4) (or of (5)). Since

$$\log q_j a_j^t \equiv \log q_j + t \log a_j,$$

the logarithm of $q_j a_j^t$ is a linear, and therefore a convex, function of t. But the class of positive functions having convex logarithms, like the class of positive functions $p(x, y)$ having subharmonic logarithms [2, p. 651], is closed under the operation of addition. Hence (4) is convex; and similarly (5) is convex.

As implied above, the increasing function $\mathfrak{M}_t(a, q)$ has two horizontal asymptotes; but it is not necessarily convex for $-\infty < t < 0$, and is not necessarily concave for $0 < t < +\infty$ [1, 6].

Though the decreasing function $\mathfrak{S}_t(a)$, $-\infty < t < 0$, has a horizontal asymptote, it is not necessarily concave; in particular, for

$$a_1 = a_2 = \cdots = a_n = a, \tag{6}$$

we find that $d^2\mathfrak{S}_t(a)/dt^2$ has exactly one zero, namely at the negative value

$$t = \log 1/\sqrt{n}. \tag{7}$$

We note that for (a) and t given by (6) and (7) respectively, we have

$$\mathfrak{S}_t(a) = a/e^2,$$

so that, in this case, *the ordinate of the inflection point is independent of n.*

One well might ask concerning bounds on the number of possible inflection points of $\mathfrak{M}_t(a, q)$ for $-\infty < t < +\infty$, and of $\mathfrak{S}_t(a)$ for $-\infty < t < 0$.

The decreasing function $\mathfrak{S}_t(a)$, $0 < t < +\infty$, has a vertical asymptote and a horizontal asymptote. It is our purpose to show that the function

$$\log \mathfrak{S}_t(a) \equiv \log \left(\sum_{j=1}^{n} a_j^t \right)^{1/t}$$

is a convex function of t for $0<t<+\infty$, and consequently that $\mathfrak{S}_t(a)$ *is a convex function of t for $0<t<+\infty$*.

2. Lemma. We shall use the following lemma.

LEMMA. *If $f(t)$ and $g(t)$ are positive non-increasing convex functions for $b<t<c$, then the function*

$$\phi(t) = f(t)\cdot g(t)$$

is convex in the interval (b, c).

Proof. Let t_1, t_2 satisfy $b<t_1<t_2<c$. Since $f(t)$ and $g(t)$ are non-increasing, we have

$$[f(t_2) - f(t_1)][g(t_1) - g(t_2)] \leqq 0,$$

or

$$f(t_1)\cdot g(t_2) + f(t_2)\cdot g(t_1) \leqq f(t_1)\cdot g(t_1) + f(t_2)\cdot g(t_2). \tag{8}$$

Since $f(t)$ and $g(t)$ are convex, we have

$$f\left(\frac{t_1 + t_2}{2}\right) \leqq \frac{1}{2}[f(t_1) + f(t_2)], \tag{9}$$

and

$$g\left(\frac{t_1 + t_2}{2}\right) \leqq \frac{1}{2}[g(t_1) + g(t_2)]. \tag{10}$$

Since $f(t)$ and $g(t)$ are positive, from (8), (9), and (10) we obtain, by multiplication,

$$\phi\left(\frac{t_1 + t_2}{2}\right) \leqq \frac{1}{2}[\phi(t_1) + \phi(t_2)],$$

so that $\phi(t)$ is convex in the interval (b, c).

3. Convexity theorems. We shall use the lemma in §2 in establishing the following result.

THEOREM 1. *The function* $\log \mathfrak{S}_t(a)$ *is convex for* $0<t+\infty$.

Proof. Let t_0 be an arbitrary positive value, and consider the functions

$$f(t) \equiv 1/t,$$

$$g(t) \equiv \log\left[\mathfrak{S}_t\left(\frac{a}{\mathfrak{S}_{t_0}(a)}\right)\right]^t,$$

for $0<t<t_0$. Clearly the hypotheses of the lemma concerning $f(t)$ are satisfied.

As observed in the Introduction, $g(t)$ is a decreasing convex function of t for $0<t<+\infty$. Since for any positive constant k we have

$$\mathfrak{S}_t(ka) = k\mathfrak{S}_t(a),$$

and since $\mathfrak{S}_t(a)$ is a decreasing function of t for $0<t<+\infty$, it follows that

$$\mathfrak{S}_t\left(\frac{a}{\mathfrak{S}_{t_0}(a)}\right) \equiv \frac{\mathfrak{S}_t(a)}{\mathfrak{S}_{t_0}(a)} > 1, \qquad (0 < t < t_0),$$

so that $g(t)$ is positive. Therefore, by the lemma, the function

$$f(t)\cdot g(t) \equiv \log \mathfrak{S}_t(a) - \log \mathfrak{S}_{t_0}(a)$$

is convex for $0<t<t_0$. Hence $\log \mathfrak{S}_t(a)$ is convex for $0<t<t_0$ and therefore for $0<t<+\infty$.

THEOREM 2. *The function $\mathfrak{S}_t(a)$ is convex for $0<t<+\infty$.*

Proof. It is well known [2, p. 651] that if a function $p(t)$ is positive and $\log p(t)$ is convex, then also $p(t)$ is convex.

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Insomnia. Again there was the procession of figures, parentheses, fractions, equations, cube and square roots. The second sleepless night was flying by imperceptibly.—Merejkowski, *The Romance of Leonardo De Vinci.*

Murder. "In order to understand these crimes," he began, "we must consider the stock-in-trade of a mathematician, for all his speculations and computations tend to emphasize the relative insignificance of this planet and the unimportance of human life."—S. S. Van Dine, *The Bishop Murder Case.*

The casual geometer. I wish he had explained why, but he merely threw out the statement with just the same take-it-or-leave-it casualness as Euclid used when he made his celebrated observation about parallel straight lines.—Somerset Maugham, *Cakes and Ale.*

Not only the Dutch. Bartoline was as dull at drawing inferences as any Dutch professor of mathematics.—Walter Scott, *Heart of Mid-Lothian.*—*E. D. Schell.*

THE MODULI OF THE ROOTS OF AN ALGEBRAIC EQUATION

ELIZABETH BEAMAN, Duke University

1. Introduction. The purpose of this paper is to discuss the polynomial $g(n)$ whose zeros are the norms of the zeros of a given polynomial $f(z)$ with complex coefficients, the norm of a number being defined as the product of the number and its complex conjugate or equivalently as the square of the modulus of the number.

For the quadratic

$$f(z) = az^2 + 2bz + c, \tag{1}$$

the explicit form of the polynomial is

$$g(n) = (Na)n^2 - 2(N\sqrt{d} + Nb)n + Nc, \tag{2}$$

where $4d$ is the discriminant $4(b^2-ac)$ and N denotes the norm. The polynomial whose roots are the moduli of the roots of $f(z)$ is

$$m(r) = |a|r^2 - (\sqrt{2}\sqrt{|b^2 - ac| + |b|^2 + |ac|})r + |c|. \tag{3}$$

Finding $g(n)$ for the cubic is reduced to the problem of finding $\sum Nz$ for the reduced cubic. The value $\sum Nz$ for the reduced cubic is one of the roots of a specified cubic.

The general case is not completely treated. The polynomial $g(n)$ for an equation of degree m is shown, however, to be a factor of a polynomial $R(n)$ whose degree is m^2. As a by-product, there is found a method for obtaining the equation of degree mp, whose roots are the products of the m roots of a polynomial $f(z)$ by the p roots of a polynomial $g(z)$.

2. The quadratic. The most direct way to find the polynomials for the quadratic case is perhaps to compute the elementary symmetric polynomials Nz_1+Nz_2, Nz_1Nz_2, and $|z_1|+|z_2|$, $|z_1|\cdot|z_2|$. The geometric theorem that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the four sides gives

$$Nz_1 + Nz_2 = \tfrac{1}{2}[N(z_1 + z_2) + N(z_1 - z_2)]. \tag{4}$$

Now

$$z_1 + z_2 = -\frac{2b}{a}, \qquad a^2(z_1 - z_2)^2 = 4d,$$

whence

$$N(z_1 + z_2) = \frac{4Nb}{Na}, \qquad N(z_1 - z_2) = \frac{4N\sqrt{d}}{Na},$$

so that

$$Nz_1 + Nz_2 = \frac{2(Nb + N\sqrt{d})}{Na}. \tag{5}$$

From

$$z_1 z_2 = \frac{c}{a}$$

we get at once

$$Nz_1 Nz_2 = \frac{Nc}{Na}.$$

Extracting the non-negative square root of the last relation we have

$$|z_1| \cdot |z_2| = \frac{|c|}{|a|}.$$

Substituting in the identity

$$(|z_1| + |z_2|)^2 = Nz_1 + 2|z_1| \cdot |z_2| + Nz_2$$

and extracting the non-negative square root we get

$$|z_1| + |z_2| = \sqrt{2}\,\frac{\sqrt{|b^2| + |ac| + |b^2 - ac|}}{|a|}.$$

Hence the coefficients in (2), (3) have the values stated.

3. The cubic. $g(n)$ for the cubic

$$f(z) = az^3 + bz^2 + cz + d$$

can be discussed as follows:

We may write

$$g(n) = (Na)n^3 - xn^2 + yn - Nd,$$

where x and y are to be determined and, in particular,

$$xNa = \phi(a, b, c, d).$$

The polynomial whose roots are the reciprocals of Nz_1, Nz_2, and Nz_3 is the polynomial $g(n)$ corresponding to

$$z^3 f\left(\frac{1}{z}\right) = dz^3 + cz^2 + bz + a,$$

that is,

$$(Nd)n^3 - yn^2 + xn - Na,$$

and

$$yNd = \phi(d, c, b, a).$$

From this,

$$\sum Nz_1Nz_2 = \frac{y}{Na} = \frac{\phi(d, c, b, a)}{Nad}.$$

Therefore, the problem is reduced to that of finding

$$\sum Nz_1 = \frac{x}{Na}.$$

Let A, B, and C be vectors representing z_1, z_2, and z_3, respectively. By computing

$$(A + B + C)^2 + (A + B - C)^2 + (A - B + C)^2 + (-A + B + C)^2,$$

we obtain the formula

$$N\sum z_1 + \sum N(-z_1 + z_2 + z_3) = 4\sum Nz_1. \tag{6}$$

By (4) we have

$$Nz_1 + N(z_2 - z_3) = \tfrac{1}{2}[N(z_1 + z_2 - z_3) + N(z_1 - z_2 + z_3)],$$

whence

$$\sum Nz_1 + \sum N(z_2 - z_3) = \sum N(-z_1 + z_2 + z_3).$$

Use of this reduces (6) to

$$\sum N(z_2 - z_3) = 3\sum Nz_1 - N\sum z_1. \tag{7}$$

The left member of this equation is invariant under the translation

$$z_1' = z_1 + h, \qquad z_2' = z_2 + h, \qquad z_3' = z_3 + h.$$

Choose h so that $\sum z_1' = 0$. This gives the reduced cubic, for which (7) becomes

$$\sum N(z_2 - z_3) = 3\sum Nz_1'.$$

Substituting in (7) gives the formula

$$\sum Nz_1 = \sum Nz_1' + \tfrac{1}{3}N\sum z_1 \tag{8}$$

which expresses $\sum Nz_1$ in terms of the sum of the roots of $f(z)$ and the sum of the norms of the roots of the reduced $f(z)$.

Accordingly, the problem of finding the norms for the general cubic has been reduced to that for the reduced cubic.

4. The reduced cubic. The polynomial

$$f(z) = z^3 + 3pz + 2q$$

has the roots*

$$t + u, \qquad \omega t + \omega^2 u, \qquad \omega^2 t + \omega u, \tag{9}$$

* J. M. Thomas, Theory of Equations, New York, 1938, p. 105.

where

$$tu = -p, \tag{10}$$

t^3, u^3 are the roots of

$$T^2 + 2qT - p^3 = 0, \tag{11}$$

and ω is a primitive cube root of unity. Let 108Δ be the discriminant of the cubic:

$$\Delta = -p^3 - q^2.$$

Direct substitution from (9) in (7) gives

$$\sum Nz_1 = 3(Nt + Nu).$$

Now

$$(Nt + Nu)^3 = Nt^3 + Nu^3 + 3NtNu(Nt + Nu). \tag{12}$$

From (10)

$$NtNu = Np.$$

Applying (5) to (11), whose roots are t^3, u^3, gives

$$Nt^3 + Nu^3 = 2(Nq + N\sqrt{-\Delta}).$$

Hence (12) can be written

$$(Nt + Nu)^3 = 2(Nq + N\sqrt{-\Delta}) + 3Np(Nt + Nu),$$

or if we write

$$V = Nt + Nu = \tfrac{1}{3}\sum Nz_1,$$

the equation for determining $\sum Nz_1$ can be taken as

$$V^3 - 3Np \cdot V - 2(Nq + N\sqrt{-\Delta}) = 0. \tag{13}$$

This cubic has real coefficients. Its resolvent quadratic, like (11), is

$$V^2 - 2(Nq + N\sqrt{-\Delta})V + Np^3 = 0, \tag{14}$$

whose discriminant is

$$4[(Nq + N\sqrt{-\Delta})^2 - Np^3]. \tag{15}$$

By computing the elementary symmetric polynomials for

$$t\bar{t} + u\bar{u}, \qquad \omega t\bar{t} + \omega^2 u\bar{u}, \qquad \omega^2 t\bar{t} + \omega u\bar{u},$$

where $\bar{t}$ denotes the complex conjugate of t, it is readily verified that these numbers are the roots of (13). These numbers can be all real only if the last two are equal. Hence the solution of the cubic (13) never falls into the so-called irreducible case.

It immediately follows that the roots of (14) are $(t\bar{t})^3$, $(u\bar{u})^3$. Since both these numbers are real, from (15) we infer the truth of the following inequality:

$$Np^{3/2} \leqq Nq + N(p^3 + q^2)^{1/2} \tag{16}$$

for all complex p, q.

5. Multiplication of roots. Consider simultaneously a polynomial $f(z)$ of degree m in the indeterminate z and a polynomial $\bar{f}(\bar{z})$ of degree p in the indeterminate $\bar{z}$, where for the moment z and $\bar{z}$, f and $\bar{f}$ are unrelated. Put

$$z = rs, \qquad \bar{z} = r/s, \qquad s \neq 0.$$

The condition that $f(rs)$ and $s^p\bar{f}(rs^{-1})$ have a root s in common is the vanishing of their resultant R, which can be shown to be a polynomial of degree $2mp$ in r. Moreover, since changing r into $-r$ can be compensated by changing s into $-s$, R is a polynomial of degree mp in r^2, or in n, if we define n as r^2. Since

$$z\bar{z} = n,$$

the roots of $R(n)$ are precisely the mp products $z_j\bar{z}_k$, where z_j is a root of $f(z)$ and $\bar{z}_k$ a root of $\bar{f}(\bar{z})$.

If now $s = e^{i\theta}$ and r is real, $\bar{z}$ is the complex conjugate of z and n is Nz. If further $\bar{f}(\bar{z})$ is made the polynomial in $\bar{z}$ whose coefficients are the complex conjugates of those of $f(z)$, then $m = p$ and $R(n)$ is a polynomial of degree m^2 in n with roots which can be grouped into m sets of m each:

$$\begin{aligned}
&z_1\bar{z}_1, \quad z_1\bar{z}_2, \cdots, z_1\bar{z}_m,\\
&z_2\bar{z}_1, \quad z_2\bar{z}_2, \cdots, z_2\bar{z}_m,\\
&\cdots\cdots\cdots\cdots\\
&z_m\bar{z}_1, \quad z_m\bar{z}_2, \cdots, z_m\bar{z}_m.
\end{aligned}$$

The numbers on the main diagonal are the norms of the roots of $f(z)$.

6. The general case. The polynomial $R(n)$ can be written

$$R(n) = g(n)h(n),$$

where $g(n)$ is the polynomial sought and $h(n)$, of degree $m(m-1)$, is a polynomial whose roots occur in $\frac{1}{2}m(m-1)$ pairs, the members of each pair being conjugate imaginary or real and equal. The factorization of $R(n)$ therefore will lead to a solution of the general problem.

The viewpoint of factorization can be successfully adopted in the quadratic and cubic cases. The explicit form of the factor $h(n)$ can readily be obtained in the quadratic case:

$$h(n) = (Na)n^2 + 2(N\sqrt{d} - Nb)n + Nc, \qquad m = 2. \tag{17}$$

It can be verified directly that the factors of this $h(n)$ are conjugate imaginary except when they are equal and that $g(n)$, $h(n)$ have a root in common only when $f(z)$ has equal roots or a zero root.

ELLIPTIC EULEROIDS

M. L. MACQUEEN AND R. W. HARTLEY, Southwestern College

1. Introduction. Let X, Y be the ordinary orthogonal cartesian coordinates of a point P on the ellipse

$$X = a\cos\theta, \qquad Y = b\sin\theta \tag{1}$$

whose foci are the points $F(c, 0)$, $F'(-c, 0)$. It will be recalled that the Euler line of a triangle contains the circumcenter, the orthocenter, and the centroid of the triangle. Let l be the Euler line of the triangle PFF'. It is the purpose of this paper to investigate the envelope of the Euler line l as the point P moves along the ellipse (1).

Parametric equations of the envelope are deduced in §2. Its general properties are considered in §3. The last section contains a brief study of four special cases in which interesting examples of algebraic curves are exhibited.

2. Elliptic euleroids. A simple calculation suffices to show that the Euler line l of the triangle PFF' has the equation

$$y - \frac{b}{3}\sin\theta = \frac{(3a^2 - b^2)\sin^2\theta - 3b^2}{ab\sin 2\theta}\left(x - \frac{a}{3}\cos\theta\right), \tag{2}$$

where x, y are the coordinates of a variable point on the line. We now apply the usual method of finding envelopes and so differentiate equation (2) partially with respect to θ. The resulting equation and equation (2) yield, on placing $\sin\theta = u$ and introducing the eccentricity e of the ellipse, the following parametric equations of the envelope of the line l, namely:

$$\begin{aligned} x &= \pm\frac{a(1-u^2)^{3/2}(1-e^2+e^2u^2)}{(7e^2-4)u^2+3(1-e^2)}, \\ y &= \frac{a}{2\sqrt{1-e^2}}\,\frac{u\left[(1-e^2)^2-2(1-e^2)(1-4e^2)u^2-e^2(e^2+2)u^4\right]}{(7e^2-4)u^2+3(1-e^2)}, \end{aligned} \tag{3}$$

where the parameter u is defined on the range $-1 \leqq u \leqq 1$, and $0 \leqq e < 1$. We propose to call the curve defined by equations (3) an *elliptic euleroid*.

3. General properties. It is evident that the character of the particular curve of the family defined by equations (3) depends upon the value of the eccentricity e. Furthermore, it is geometrically obvious that euleroids possess symmetry properties which are identical with those of the ellipse.

We readily verify that every euleroid meets the x-axis in the points $(\pm a/3, 0)$ for which $u = 0$. With the exception of the curve obtained when $e = \frac{1}{2}$, all euleroids have two double points on the x-axis. The parameter values for these points are the two real roots of the equation

$$e^2(e^2+2)u^4 + 2(1-e^2)(1-4e^2)u^2 - (1-e^2)^2 = 0. \tag{4}$$

The only intersections with the y-axis are obtained when $u = \pm 1$, so that these points have the coordinates

$$\left(0, \pm \frac{a(1 - 2e^2)}{2\sqrt{1 - e^2}}\right). \tag{5}$$

From equations (3) we find

$$\frac{dy}{dx} = \pm \frac{1}{2\sqrt{1 - e^2}} \frac{(e^2 + 2)u^2 - 3(1 - e^2)}{u(1 - u^2)^{1/2}}. \tag{6}$$

Thus the curve has horizontal tangents at points whose parameter values are obtained from the equation

$$u = \pm \sqrt{3} \left(\frac{1 - e^2}{2 + e^2}\right)^{1/2}. \tag{7}$$

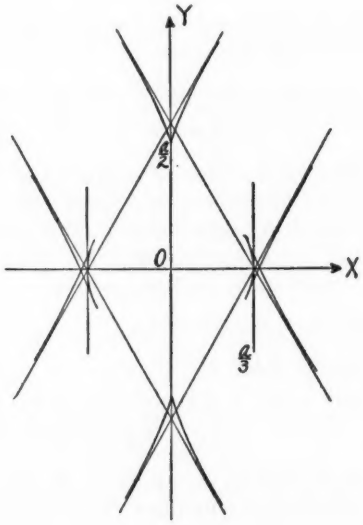

FIG. 1.

Since $|u| \leqq 1$, it is an easy matter to show that no horizontal tangents exist when $e < \frac{1}{2}$. However, when $\frac{1}{2} \leqq e < 1$, points always exist at which the curve has horizontal tangents. The only euleroid with horizontal tangents at the points where $u = \pm 1$ is the curve with $e = \frac{1}{2}$.

All euleroids have the lines $x = \pm a/3$ for vertical tangents at the points where $u = 0$. The y-axis is also a vertical tangent, except when $e = \frac{1}{2}$.

The existence of real asymptotes depends upon parameter values which satisfy the equation

$$u = \pm \sqrt{3}\left(\frac{e^2 - 1}{7e^2 - 4}\right)^{1/2}. \tag{8}$$

This equation enables us to show that all euleroids for which $e<\frac{1}{2}$ possess real asymptotes. No real asymptotes exist when $\frac{1}{2}\leqq e<1$. When $e<\frac{1}{2}$, we find that there are four real asymptotes which are parallel in pairs (see Fig. 1). The equations of the asymptotes are found to be

$$\pm \sqrt{3}\,(1 - 4e^2)^{1/2}(4 - 7e^2)^{1/2}x \pm (4 - 7e^2)^{1/2}y = a(2 - 5e^2)/\sqrt{3}. \tag{9}$$

It is evident from (9) that the slopes of the two pairs of asymptotes are given by the expression $\pm\sqrt{3}(1-4e^2)^{1/2}$, which is imaginary when $e>\frac{1}{2}$.

It is of some interest to follow the change in the position of the asymptotes as e varies from $e=0$ to $e=\frac{1}{2}$. The maximum absolute value of the slopes of the asymptotes is $\sqrt{3}$, and occurs when $e=0$. The asymptotes intersect orthogonally (so that their slopes are ± 1) when $e=\sqrt{6}/6$. Finally, when $e\to\frac{1}{2}$, the limiting value of the slopes is zero.

The expression for the y-intercepts of the asymptotes may be obtained immediately from equation (9). From this result we readily find that the maximum absolute value of the y-intercepts is $a/\sqrt{3}$, occurring when $e=0$. Moreover, when e increases in value the y-intercepts decrease in numerical value but approach $a\sqrt{3}/6$ as a limit, when $e\to\frac{1}{2}$.

It may be remarked, in passing, that y is an integral rational function of the parameter u in the cases $e=0$, $e=\frac{1}{2}$, and $e=2/\sqrt{7}$.

In general, when the parameter is eliminated from equations (3), we obtain an algebraic equation of the tenth degree in which the term involving x^{10} is given by

$$576e^8(e^2 + 2)^8(1 - 4e^2)^4(1 - e^2)^4x^{10}. \tag{10}$$

The coefficient of this term vanishes, so that the degree is less than ten if, and only if, $e=0$ or $e=\frac{1}{2}$. In each of these two cases we later prove that the euleroid is an algebraic curve of order six.

4. Special cases. In this section we consider four special cases which appear to be of some interest.

(a) *The case when* $e=0$. It follows at once that equations (3) reduce to

$$x = \pm \frac{a(1 - u^2)^{3/2}}{3 - 4u^2}, \qquad y = \frac{au(1 - 2u^2)}{2(3 - 4u^2)}. \tag{11}$$

This euleroid meets the x-axis in the points $(\pm a/3, 0)$ for which $u=0$, and at these points the curve has vertical tangents. The points $(\pm a\sqrt{2}/4, 0)$ are double points which correspond to the parameter values $u=\pm 1/\sqrt{2}$. The slopes of the tangents to the curve at each of the double points are ± 2, as may be verified by means of the derivative.

$$\frac{dy}{dx} = \pm \frac{2u^2 - 3}{2u(1 - u^2)^{1/2}}. \tag{12}$$

Incidentally, it may be remarked that the absolute value of the derivative in (12) is always greater than unity.

The intersections with the y-axis occur at the points $(0, \pm a/2)$ which are cusps on the curve, the y-axis being the common cuspidal tangent. The curve also has four cusps at the points $(\pm 3a\sqrt{3}/16, \pm a/16)$ where the parameter values are $u = \pm \frac{1}{2}$. To demonstrate the existence of these four cusps it will be sufficient to consider the one which lies in the first quadrant. From equations (11) we find

$$dx/du = N(1 - 2u), \qquad dy/du = P(1 - 2u), \tag{13}$$

where we have placed

$$N = -\frac{au(1 + 2u)(1 - u^2)^{1/2}}{(3 - 4u^2)^2}, \qquad P = \frac{a(1 + 2u)(3 - 2u^2)}{(3 - 4u^2)^2}. \tag{14}$$

When u increases from $u = 0$ to $u = \frac{1}{2}$, the signs of the derivatives in (13) show that y increases while x decreases. Both derivatives in (13) vanish when $u = \frac{1}{2}$. Moreover, when $u > \frac{1}{2}$, y decreases while x increases. Since the slope of the tangent to the curve at the point where $u = \frac{1}{2}$ is $-5/\sqrt{3}$, we conclude from these results that the curve has a cusp at the point in the first quadrant where $u = \frac{1}{2}$.

The curve has four real asymptotes whose equations are obtained by placing $e = 0$ in equation (9). The resulting equations are

$$y = \pm \sqrt{3}\, x \pm a/\sqrt{3}. \tag{15}$$

Elimination of the parameter from equations (11) yields the following algebraic equation of the sixth degree:

$$64(x^2 + y^2)(3x^2 - y^2)^2 - 208a^2x^4 - 128a^2x^2y^2 - 48a^2y^4 + 25a^4x^2 + 12a^4y^2 - a^6 = 0. \tag{16}$$

The resulting graph is sketched in Fig. 1.

(b) *The case when* $e = \frac{1}{2}$. Placing $e = \frac{1}{2}$ in equations (3) shows that the equations of the euleroid in this case are

$$x = \pm (a/9)(u^2 + 3)(1 - u^2)^{1/2}, \qquad y = (\sqrt{3}\, au/12)(1 + u^2). \tag{17}$$

With the aid of the derivative

$$\frac{dy}{dx} = \pm \frac{3\sqrt{3}}{4} \frac{(1 - u^2)^{1/2}}{u}, \tag{18}$$

we readily determine the nature of the graph. The curve has vertical tangents at the points $(\pm a/3, 0)$ where it meets the x-axis. Furthermore, we find that the

points $(0, a\sqrt{3}/6)$, $(0, -a\sqrt{3}/6)$ are respectively maximum and minimum points with horizontal tangents to the curve at these points.

The result of eliminating the parameter from equations (17) shows that this euleroid is an algebraic curve of the sixth order represented by the equation

$$3(27x^2 + 16y^2)^3 - 18{,}225a^2x^4 - 13{,}834a^2x^2y^2 + 5{,}120a^2y^4 + 1{,}872a^4x^2 + 256a^4y^2 - 64a^6 = 0. \tag{19}$$

The character of the graph is shown in Fig. 2.

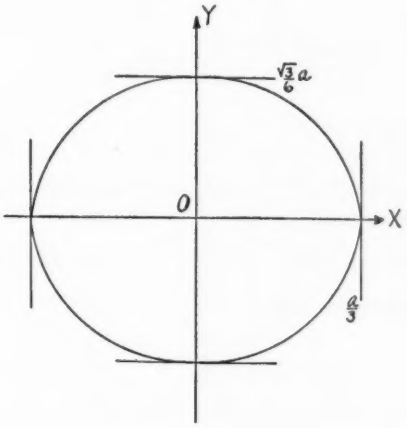

FIG. 2.

(c) *The case when* $e = 1/\sqrt{2}$. The parametric equations of the curve in this case are found to be

$$x = \pm \frac{a(1+u^2)(1-u^2)^{3/2}}{3-u^2}, \qquad y = \frac{\sqrt{2}\,a}{4}\,\frac{u(1+5u^2)(1-u^2)}{3-u^2}. \tag{20}$$

From these equations we find

$$\frac{dy}{dx} = \pm \frac{\sqrt{2}}{4}\,\frac{5u^2-3}{u(1-u^2)^{1/2}}, \tag{21}$$

so that this euleroid not only has vertical tangents at the points $(\pm a/3, 0)$, but also a vertical tangent at the origin. In fact, we readily verify that the curve has two cusps at the origin, the y-axis being the common cusp tangent. Next we find that horizontal tangents exist at points of the curve where the parameter values are $u = \pm\sqrt{15}/5$. The points with coordinates $(\pm 4a\sqrt{10}/75,\ a/\sqrt{30})$ are maximum points, while the points $(\pm 4a\sqrt{10}/75,\ -a/\sqrt{30})$ are minimum points.

Elimination of the parameter from equations (20) yields an algebraic equation of the tenth degree, but we shall not write this equation here.

The character of the graph is indicated in Fig. 3.

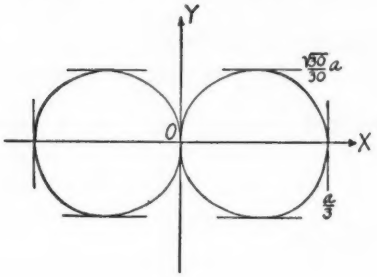

FIG. 3.

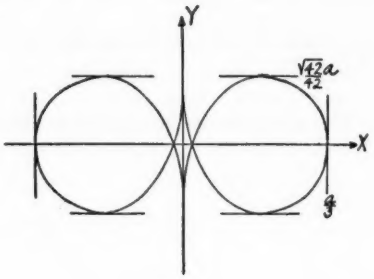

FIG. 4.

(d) *The case when* $e=2/\sqrt{7}$. In this case the euleroid is represented parametrically by the equations

$$x = \pm\,(a/9)(3 + 4u^2)(1 - u^2)^{3/2}, \qquad y = (au\sqrt{21}/42)(1 + 6u^2 - 8u^4). \tag{22}$$

From these equations we find

$$\frac{dy}{dx} = \mp\,\frac{3\sqrt{21}}{14}\,\frac{(1 - 2u^2)}{u(1 - u^2)^{1/2}}. \tag{23}$$

The curve meets the x-axis in the points $(\pm a/3,\ 0)$ where it has vertical tangents. Moreover, the curve has two double points on the x-axis at which the parameter values are $u=\pm\frac{1}{4}(6+2\sqrt{17})^{1/2}$. The coordinates of the double points are found to be

$$\left(\pm\,\frac{(71 - 17\sqrt{17})^{1/2}}{18}\,a,\ 0\right).$$

The slopes of the double point tangents are $\pm 3/\sqrt{42}/14(\sqrt{17}-1)^{1/2}$.

This curve meets the y-axis in the points $(0,\ \pm a\sqrt{21}/42)$, and at each of these points the curve has a cusp with the y-axis as the common cusp tangent.

The curve has horizontal tangents at the points where the parameter values are $u=\pm 1\sqrt{2}$. We find that the points $(\pm 5a\sqrt{2}/36,\ a/\sqrt{42})$ are maximum points, and that the points $(\pm 5a\sqrt{2}/36,\ -a/\sqrt{42})$ are mimimum points on the curve.

If the parameter is eliminated from equations (22), an algebraic equation of the tenth degree is obtained, but we shall omit the writing of this result.

This analysis enables us to sketch the graph indicated in Fig. 4.

A GEOMETRIC APPROACH TO THE COVARIANTS OF A CUBIC*

A. E. HALTEMAN, University of Wisconsin

1. The covariants. The fundamental covariants of a binary cubic, $a_0x^3+a_1x^2y+a_2xy^2+a_3y^3$, are usually obtained algebraically [1]. If the form is equated to zero the three roots of the resulting cubic equation in x and y can be represented by three points A, B, C, of the complex plane. The cross ratio of these three points and a fourth point D is written as (AB, CD) and defined by

$$\alpha = (AB, CD) = \frac{z_3 - z_1}{z_3 - z_2}\cdot\frac{z_4 - z_2}{z_4 - z_1}, \tag{1}$$

where the coordinates of A, B, C, D, are z_1, z_2, z_3, z_4, respectively. If $\alpha=a+bi$ is given and the points A, B, C, are fixed, and distinct, then the position of D is uniquely determined by (1).

If three points A, B, C, and a value of α are given, then six points $A' \cdots C''$ are determined by $(AB, CC')=\alpha$, $(BC, AA')=\alpha$, $(CA, BB')=\alpha$ $(BA, CC'')=\alpha$, $(AC, BB'')=\alpha$, $(CB, AA'')=\alpha$. The triads $A'\ B'\ C'$, $A''\ B''\ C''$ are covariants of the triad ABC for each α, since a linear functional transformation

$$w = \frac{az + b}{cz + d} \qquad ad - bc \neq 0 \tag{2}$$

which transforms a given triad ABC into $A_1B_1C_1$ will transform $A'B'C'$ and $A''B''C''$ into $A_1'B_1'C_1'$ and $A_1''B_1''C_1''$ with $(A_1B_1, C_1C_1')=\alpha$, *etc.*

In this article are developed geometric relations between the triads ABC, $A'B'C'$, and $A''B''C''$ where α is restricted to $|\alpha|=1$. The general relations are developed by a study of the relations which exist for a particular triad ABC and then we use the linear fractional transformations (2) and the properties which they possess [2, Chap. V].

In the final section we obtain the equation of the cubic whose roots are the coordinates of A', B', C', in terms of the coefficients of the cubic whose roots are A, B, C, and α, where α is any complex number.

2. Principal theorems. In this section we present a number of theorems.

THEOREM 1. *Any three distinct points A, B, C, in the complex plane together with a cross ratio α, where α is a complex number such that $|\alpha|=1$, determine two triads of points $A'B'C'$, $A''B''C''$ such that a point of either of the triads is the inverse of the corresponding point to the other triad with respect to the circle determined by A, B, and C.*

* This paper is part of a thesis, written under the direction of Professor F. E. Woodard, presented in partial fulfillment of the requirements for the Master of Arts degree at the University of Oregon.

In the special values $A \equiv -1$, $B \equiv 1$, $C \equiv k$, where k is any real number, and $\alpha = a + bi$, lead to values for A' and for A''. Use of the fact that $|\alpha| = 1$ makes $A' = \overline{A}''$, where $\overline{A}$ is the conjugate imaginary of A. The linear fractional transformation will take the axis of reals into a circle. A' and A'' will be inverse points with respect to the circle ABC. This property is maintained by the linear fractional transformation, so that the theorem is true for any three distinct points A, B, C.

THEOREM 2. *The locus of the points A' of Theorem 1, as α varies under the restriction $|\alpha| = 1$, is a circle orthogonal to the circle ABC at the point A.*

In the proof of Theorem 1 let us take $k = 0$ and obtain an expression for A' in terms of a and b. A' is a complex number which we may represent by $A' = x + yi$. If we equate the real parts and the imaginary parts we then obtain two equations from which a and b can be eliminated. The result is the equation of a circle with center on the axis of reals passing through the point A. Therefore the circle upon which A' moves is orthogonal to the circle ABC (the axis of reals) at the point A. Using the linear fractional transformation the result may be generalized to prove the theorem. Similarly the circle on which B' moves and the circle on which C' moves may be determined.

COROLLARY. *Both A' and A'' move on a single circle; as do B' and B'', C' and C''.*

A'' moves on a circle inverse to the circle on which A' moves. The circles coincide since an orthogonal circle is its own inverse.

The cross ratio of the four points A, C, A', C', in the particular case $(-1\ 1,\ 0C') = (1\ 0,\ -1A') = a + bi$ is real. Therefore the four points lie on a circle. In general $ACA'C'$, $BCB'C'$, and $ABA'B'$, lie on circles as do $ACA''C''$, $BCB''C''$, and $ABA''B''$.

THEOREM 3. *If θ is the positive acute angle between the circle ABC and the circle $ACA'C'$ we have $\theta = \arctan |b/1-a|$.*

The equation of the circle through $-1, 0, A'$, is

$$x^2 + y^2 + x + \frac{1-a}{b} y = 0. \tag{3}$$

The circle (3) cuts the circle ABC (the axis of reals) at -1 and 0. The slope of the tangent at these points is $dy/dx = b/1-a$ which in this case gives the tangent of the angle between the two curves. To obtain the positive acute angle we use $|dy/dx|$. The linear fractional transformation preserves the angle between the two curves; thus the theorem is true in general.

THEOREM 4. *There are six circles making an angle θ with circle ABC, four through each of the points A, B, C. These circles are tangent in pairs at the points A, B, C, or intersect by pairs in the points $A' \cdots C''$, provided that $\alpha \neq \pm 1$.*

At a point, say C, there are two lines making an angle θ with circle ABC. Circle $ACA'C'$ is tangent to one of these lines. Of the other two circles through C but not through A, one is tangent to circle $ACA'C'$ at C. The other has the two points C and C' in common with it.

THEOREM 5. *When* $\alpha=-1$, $A'\equiv A''$, $B'\equiv B''$, $C'\equiv C''$, *and the three points lie on the circle* ABC.

From Theorem 3 we have $\theta=0$. The six circles of Theorem 4 are identical with circle ABC. $(BC, AA')=\alpha$, $(CB, AA')=1/\alpha$, $(CB, AA'')=\alpha$; but $\alpha=-1$, so $A'\equiv A''$.

Let us take $\alpha=\omega_1$, where ω_1 is an imaginary cube root of minus one. We define the Hessian points of the triad ABC, H_1 and H_2, by $(AB, CH_1)=\omega_1$,, $(BA, CH_2)=\omega_1$. (Algebraically the triad ABC may be considered as the roots of a cubic equation. Each of the points H_1 and H_2 is known [1, p. 197] to form an equianharmonic range with the triad ABC.) It can be shown for $\alpha=\omega_1$ that $A'\equiv B'\equiv C'\equiv H_1$ and $A''\equiv B''\equiv C''\equiv H_2$; the Hessian points are inverse points with respect to circle ABC.

THEOREM 6. *The Hessian points,* H_1 *and* H_2, *lie on circles through* AB, BC, CA, *making angles of* $60°$ *with the circle* ABC, *and conversely.*

This theorem is an immediate consequence of Theorem 4.

THEOREM 7. *There is an infinite number of triads on the circle* ABC *having the Hessian points* H_1, *and* H_2.

Take a fixed circle O with a pair of inverse points H_1 and H_2. A circle Q through H_1 and H_2 cuts circle O in two points, say A and A_1. Two circles through A and H_1 each making an angle of $60°$ with circle O at A will cut circle O at B and at C. ABC so determined will have Hessian points H_1 and H_2 since the conditions of Theorem 6 are satisfied. There is an infinite number of circles through H_1 and H_2. We see that there is an infinite number of triads on circle O having Hessian points H_1 and H_2.

From Theorem 2 a construction for the Hessian points can be easily indicated. We know that when $\alpha=\omega_1$, ω_2 then $A'\equiv B'\equiv C'\equiv H_1$ and $A''\equiv B''\equiv C''\equiv H_2$. The loci of Theorem 2 have two points in common, the Hessian points.

To obtain the locus of A' as α varies we use: (i) $\alpha=1$, then $A'\equiv A$; (ii) $\alpha=-1$ then $A'\equiv A_1$ which can readily be constructed; (iii) the orthogonal condition. Similarly the loci of B' and C' may be obtained.

We state the following lemma without proof.

LEMMA. *If we have four distinct points* A, B, C, C_1, *such that* $(AB, CC_1)=-1$) *then* C *and* C_1 *are inverse points with respect to the circle which meets the circle* ABC *orthogonally at* A *and* B.

The lemma leads to constructions for the fourth harmonic point to three given distinct points in the complex plane. When A, B, and C are not collinear

draw circle ABC. Construct the tangent at A and at B. They intersect at a point O. If we draw CO it will cut circle ABC in C_1. When ABC are collinear draw circle ABC, a straight line, and construct a circle S orthogonal to circle ABC at A and B. If we construct a circle S' orthogonal to circle S and orthogonal to circle ABC at C, it will cut circle ABC again at C_1. The second construction also applies if A, B, C, lie on a circle. The circle S' is easily obtained if we use its center as O' at the intersection of the line AB and the tangent to the circle ABC at C'.

The construction for the Hessian points described here is more easily indicated than the construction given by Grace and Young [1, p. 211]; however in both constructions the same circles are used.

3. Algebraic considerations. In this section we regard the triad ABC as the roots r, s, t, *of a cubic equation*

$$a_0z^3 + a_1z^2 + a_2z + a_3 = 0. \tag{4}$$

Using the cross ratio α, any complex number, and taking the roots in cyclic order, $(rs, tt') = \alpha$, $(st, rr') = \alpha$, $(tr, ss') = \alpha$ we obtain the roots r', s', t', of a cubic equation

$$A_0z^3 + A_1z^2 + A_2z + A_3 = 0. \tag{5}$$

The roots r', s', t', are cyclic symmetric functions of r, s, and t. By use of the elementary symmetric functions we obtain (5) in the form

$$(Pz^3+Qz^2+Rz+S)(\alpha^2-\alpha) \mp (2\alpha^3-3\alpha^2-3\alpha+2)(a_0z^3+a_1z^2+a_2z+a_3)\sqrt{\Delta}=0. \tag{6}$$

where

$$\begin{aligned}
P &= 2a_1^3 - 9a_0a_1a_2 + 27a_0a_3 \\
Q &= 2a_1^2a_2 + 27a_0a_1a_3 - 18a_0a_2^2 \\
R &= 18a_1^2a_3 - 3a_1a_2^2 - 27a_0a_2a_3 \\
S &= 9a_1a_2a_3 - 27a_0a_3^2 - 2a_2^3 \\
\Delta &= 18a_0a_1a_2a_3 - 4a_0a_2^3 + a_2^2a_1^2 - 4a_1^3a_3 - 27a_3^2a_0^2.
\end{aligned}$$

The ambiguity of sign in (6) will give either the cubic with roots r', s', t', or with roots r'', s'', t''.

By consideration of (6) we note that the associated cubic with roots r', s', t', is a combination of the covariant cubic [3, p. 175] and the original cubic. The covariant cubic is obtained when $\alpha = -1, 1/2, 2$, the three values gotten if and only if the four elements forming the cross ratio are a harmonic set.

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1. J. H. Grace and A. Young, Algebra of Invariants, Cambridge University press, 1903.
2. E. J. Townsend, Functions of a Complex Variable, New York, Henry Holt and Co., 1915.
3. G. Salmon, Lessons in Higher Algebra, Dublin, Hodges, Foster and Co., 1876.

DISCUSSIONS AND NOTES

EDITED BY E. F. BECKENBACH, University of California, Los Angeles 24, Calif.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE EXACT NUMBER OF PRIMES BELOW A GIVEN LIMIT*

ALFRED BRAUER, University of North Carolina

1. *The problem.* In the solution of Problem 4083 (this MONTHLY, vol. 51, 1944, pp. 479–480) the following question is raised: How can we prove that there are 50,847,478 primes less than one billion although these primes are not individually known?

The prime number theorem gives an analytic formula for the approximate number of primes which are less than or equal to a given limit x. But it does not enable us to determine the exact number $\pi(x)$ of such primes.

Let $p_1 < p_2 < \cdots < p_n$ be the primes less than or equal to $x^{1/2}$. Then

$$\pi(x) = [x] + n - 1 - \sum_{\kappa}\left[\frac{x}{p_\kappa}\right] + \sum_{\kappa<\lambda}\left[\frac{x}{p_\kappa p_\lambda}\right] - \cdots$$

where $p_\kappa, p_\lambda, \cdots$ run over the primes $p_1, p_2 \cdots, p_n$. This follows at once from a formula already given by Legendre. But for large values of x this formula is completely impracticable.

2. *A solution.* The problem in question was solved by E. Meissel in 3 papers.† In the first paper, he proved the following theorem which enables us to find $\pi(x)$ if the primes $q \leqq x^{1/2}$ are known.

Let $\phi(x, r)$ be the number of positive integers less than or equal to x which are relatively prime to each of the first r primes. Set $\pi(x^{1/3}) = m$, $\pi(x^{1/2}) = n$, and $n - m = s$. Then

$$(1) \qquad \pi(x) = \phi(x, m) + m(s + 1) + \frac{s(s-1)}{2} - 1 - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right).$$

Using (1) Meissel actually calculated $\pi(20{,}000)$, $\pi(500{,}000)$, $\pi(10^6)$, $\pi(10^7)$, $\pi(10^8)$, and $\pi(10^9)$.

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* Presented at the meeting of the Southeastern Section of the Mathematical Association of America, Raleigh, April 19, 1946.

† Über die Bestimmung der Primzahlmenge innerhalb gegebener Grenzen, Mathematische Annalen, vol. 2, 1870, pp. 636–642.—Berechnung der Menge von Primzahlen, welche innerhalb der ersten Hundert Millionen natürlicher Zahlen vorkommen, Mathematische Annalen, vol. 3,1871, pp. 523–525.—Berechnung der Menge von Primzahlen, welche innerhalb der ersten Milliarde natürlicher Zahlen vorkommen, Mathematische Annalen vol. 25, 1885, pp. 251–257.

Later other proofs for (1) were given and similar formulas were proved by A. Lugli,* F. Rogel,† and A. Baranowski.‡ But (1) remains the simplest formula for the calculation of $\pi(x)$. It was also used to find errors in the tables of primes.§

Meissel's method is outlined in a number of books, for instance in Mathews, *Theory of Numbers*, 1892, pp. 273–278, Uspenski and Heaslet, *Elementary Number Theory* 1939, pp. 117–125, Wertheim, *Elemente der Zahlentheorie*, 1887, pp. 21–25, and Wertheim, *Anfangsgründe der Zahlenlehre*, 1902, pp. 56–63.

The following very short and simple proof of (1) may be of interest.

We have $\phi(x, k-1)$ integers t which satisfy $t \leqq x$ and $(t, p_1p_2 \cdots p_{k-1}) = 1$; exactly $\phi[x/(p_k), k-1]$ of them are divisible by p_k. Hence

$$\phi(x, k) = \phi(x, k-1) - \phi\left(\frac{x}{p_k}, k-1\right). \tag{2}$$

Adding (2) for $k = m+1, m+2, \cdots, n$, we obtain

$$\phi(x, n) = \phi(x, m) - \sum_{\sigma=1}^{s} \phi\left(\frac{x}{p_{m+\sigma}}, m+\sigma-1\right). \tag{3}$$

From the definition of m, n, and s it follows that

$$x^{1/3} < p_{m+\sigma} \leqq x^{1/2} \leqq \frac{x}{p_{m+\sigma}} < x^{2/3} \qquad (\sigma = 1, 2, \cdots, s). \tag{4}$$

Now, let a and b be numbers such that $a \leqq b \leqq a^2$. The integers less than or equal to b and relatively prime to each prime less than or equal to a are the number 1 and the $\pi(b) - \pi(a)$ primes q for which $a < q \leqq b$. Applying this for $b = x$ and $a = x^{1/2}$, we obtain

$$\phi(x, n) = 1 + \pi(x) - \pi(x^{1/2}) = \pi(x) - n + 1. \tag{5}$$

For $b = x/p_{m+\sigma}$ and $a = p_{m+\sigma} - \epsilon$ where $0 < \epsilon < \min(1, p_{m+\sigma} - x^{1/3})$, we obtain, by (4),

$$\begin{aligned} \phi\left(\frac{x}{p_{m+\sigma}}, m+\sigma-1\right) &= 1 + \pi\left(\frac{x}{p_{m+\sigma}}\right) - \pi(p_{m+\sigma} - 1) \\ &= \pi\left(\frac{x}{p_{m+\sigma}}\right) - (m+\sigma-2). \end{aligned} \tag{6}$$

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* Sul numero dei numeri primi da 1 ad n, Giornale di Matematiche, vol. 26, 1888, pp. 86–95.

† Zur Bestimmung der Anzahl Primzahlen unter gegebenen Grenzen, Mathematische Annalen, vol. 36, 1890, pp. 304–315.—Die Bestimmung der Anzahl der unter einer gegebenen Grenze liegenden Primzahlen, Archiv der Mathematik und Physik (2), vol. 17, 1900, pp. 225–237.

‡ O wzorach sluzacych do oblizenia liczby liczb pierwszych nie przekraczajacych danej granicy. Rozprawy Akademii Umiejetnosci w Krakowie (2), vol. 8, 1895, pp. 192–219. (Extract in German in Anzeiger der Akademie der Wissenschaften in Krakau (1894), pp, 280–281.

§ J.-P. Gram, Rapport sur quelques calculs entrepris par M. Bertelsen et concernant les nombres premiers, Acta Mathematica, vol. 17, 1893, pp. 301–314.

It follows from (5), (3) and (6) that

$$\pi(x) = \phi(x, n) + n - 1 = \phi(x, m) - \sum_{\sigma=1}^{s} \phi\left(\frac{x}{p_{m+\sigma}}, m + \sigma - 1\right) + m + s - 1$$

$$= \phi(x, m) - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right) + \sum_{\sigma=1}^{s} (m + \sigma - 2) + m + s - 1$$

$$= \phi(x, m) + m(s+1) + \frac{s(s-1)}{2} - 1 - \sum_{\sigma=1}^{s} \pi\left(\frac{x}{p_{m+\sigma}}\right).$$

This proves (1).

The most difficult step in the use of (1) is the calculation of $\phi(x, m)$. For this calculation we may use (2) and the simple fact that for $[x] = gp_1p_2 \cdots p_k \pm r$ where g and r are non-negative integers, we have

$$\phi(x, k) = g \cdot \phi(p_1p_2 \cdots p_k) \pm \phi(r, k). \tag{7}$$

This holds since each complete system of residues (mod $p_1p_2 \cdots p_k$) contains $\phi(p_1p_2 \cdots p_k)$ integers relatively prime to the first k primes, and since with z also $gp_1p_2 \cdots p_k \pm z$ is relatively prime to $p_1p_2 \cdots p_k$. For the positive sign this was already used by Meissel, for the negative sign by Hossfeld.* Moreover, we may use

$$\phi(p_2p_3 \cdots p_k, k) = \tfrac{1}{2}\phi(p_1p_2 \cdots p_k, k) = \tfrac{1}{2}\phi(p_1p_2 \cdots p_k). \tag{8}$$

3. *An example.* As an example, let us calculate $\pi(2000)$. Since $[2000^{1/3}] = 12$, $[2000^{1/2}] = 44$, we have $m = 5$, $n = 14$, and $s = 9$, hence

$$\pi(2000) = \phi(2000, 5) + 50 + 36 - 1 - \sum_{\sigma=1}^{9} \pi\left(\frac{2000}{p_{5+\sigma}}\right).$$

Now $\pi(2000/13) = 36$, $\pi(2000/17) = 30$, $\pi(2000/19) = 27$, $\pi(2000/23) = 23$, $\pi(2000/29) = 19$, $\pi(2000/31) = 18$, $\pi(2000/37) = 16$, $\pi(2000/41) = 15$, $\pi(2000/43) = 14$, hence $\pi(2000) = \phi(2000, 5) - 113$.

The computation of $\phi(2000, 5)$ will become simpler than that of Wertheim in his book, *Anfangsgründe der Zahlenlehre*, p. 57, by the use of (7) and (8). We have

$$\phi(2000, 5) = \phi(2310) - \phi(310, 5) = 480 - \phi(310, 5),$$

$$\phi(310, 5) = \phi(310, 4) - \phi(28, 4) = \phi(210) + \phi(100, 4) - 6 = 42 + \phi(100, 4),$$

$$\phi(100, 4) = \phi(105, 4) - 2 = \tfrac{1}{2}\phi(210) - 2 = 22,$$

$$\phi(2000, 5) = 480 - 42 - 22 = 416,$$

$$\pi(2000) = 416 - 113 = 303.$$

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* Bemerkungen über eine zahlentheoretische Formel, Zeitschrift für Mathematik und Physik, vol. 35, 1890, pp. 382–384.

ON THE ORTHOPOLE

R. GOORMAGHTIGH, Bruges, Belgium

1. In a paper published recently in this MONTHLY, vol. 53, 1946, p. 200, I have generalized the following theorem considered by Murnaghan, Mathesis, vol. 41, 1927, p. 27; Godeau, Mathesis, vol. 41, 1927, p. 72; and Ramler, this MONTHLY, vol. 47, 1940, p. 140:

When two triangles inscribed in the same circle are such that the sides of the first triangle have the same orthopole as to the second triangle, the sides of the second one have also the same point as common orthopole with respect to the first triangle; and that common orthopole is the mid-point of the segment joining the orthocenters of the two triangles.

The theorem had already been given in the following more general form, different from the one I have mentioned, by Kantor, Mathesis, vol. 26, 1906, p. 144:

If two triangles $A_1A_2A_3$, $B_1B_2B_3$ are inscribed in a circle, the orthopoles A_1', A_2', A_3' of the sides of $A_1A_2A_3$, with respect to the triangle $B_1B_2B_3$, and the orthopoles B_1', B_2', B_3' of the sides of $B_1B_2B_3$, with respect to the triangle $A_1A_2A_3$, are six points on a circle having as center the mid-point of the segment joining the orthocenters of the triangles.

Absolonne has proved, Mathesis, vol. 27, 1907, p. 23, that, R being the radius of the common circumcircle and θ the angle formed by the homologous sides of $A_1A_2A_3$ and $A_1'A_2'A_3'$, $B_1B_2B_3$ and $B_1'B_2'B_3'$, the radius of the circle containing the six orthopoles is $R\cos\theta$.

2. In a recent article published in the Bulletin de la Société royale des Sciences de Liége, 1946, p. 119. I have mentioned the notion of orthopole of an $(n-1)$-sided polygon inscribed in a circle as to an n-sided polygon inscribed in the same circle, and given several theorems, one of which generalizes the just recalled Kantor theorem.

It is known that the orthopole of a chord B_1B_2 of a circle Γ as to the inscribed triangle $A_1A_2A_3$ is the intersection of the Simson lines of B_1 and B_2; when a chord B_1B_2 and a quadrilateral $A_1A_2A_3A_4$ incribed in Γ are given, the orthopoles of B_1B_2 as to the four triangles obtained by omitting one vertex in $A_1A_2A_3A_4$ are on a straight line, which we call the orthopolar line of B_1B_2 as to $A_1A_2A_3A_4$; when a triangle $B_1B_2B_3$ and the quadrilateral $A_1A_2A_3A_4$ are given, both inscribed in Γ, the orthopolar lines of the chords forming $B_1B_2B_3$ as to $A_1A_2A_3A_4$ meet at a point, which we call the orthopole of $B_1B_2B_3$ as to $A_1A_2A_3A_4$; continuing the process we get to the notion of orthopole of an $(n-1)$-sided polygon as to an n-sided one, both inscribed in Γ.

Further, to *any* n-sided polygon $A_1A_2\cdots A_n$ inscribed in Γ may be associated an n-sided *regular* polygon $L_1L_2\cdots L_n$ as follows: Ω being any point on Γ, any vertex L_i is given by

$$\Omega L_i = \frac{1}{n}(\Omega A_1 + \Omega A_2 + \cdots + \Omega A_n),$$

multiples of 2π being added to the various arcs considered. The polygon $L_1L_2 \cdots L_n$ is the same for any position of Ω on Γ and may be called the *mean* polygon of $A_1A_2 \cdots A_n$.

It may finally be noted that, if O is the center of Γ, we may call *orthocenter* of the cyclic polygon $A_1A_2 \cdots A_n$ the extremity of the resultant of the forces represented by $OA_1, OA_2 \cdots, OA_n$.

Then the generalization of Kantor's theorem which I have given is as follows:

When two n-sided polygons $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ are inscribed in a circle Γ, the n orthopoles, with respect to $A_1A_2 \cdots A_n$, of all the polygons obtained by omitting one of the vertices of $B_1B_2 \cdots B_n$ and the n orthopoles, with respect to $B_1B_2 \cdots B_n$, of all the polygons obtained by omitting one of the vertices of $A_1A_2 \cdots A_n$ are 2n points on a circle having as center the mid-point of the segment joining the orthocenters of the polygons. If ϕ is the angle formed by the mean polygons of the given polygons, the radius of the circle equals that of Γ multiplied by cos $\frac{1}{2}n\phi$ *when n is even and by* sin $\frac{1}{2}n\phi$ *when n is odd.*

FERMAT'S THEOREM AS A PROBLEM IN PROBABILITY

E. C. Molina, East Orange, New Jersey

Consider an urn containing a white and b black balls, where a and b are both positive integers. Let n drawings be made, the ball being replaced in the urn after each drawing. Then

$$P = \left(\frac{a}{a+b}\right)^n + \left(\frac{b}{a+b}\right)^n$$

is the probability that the balls drawn will be *all white or all black.*

Now let the balls be repainted so that A are white and B black, where

$$A + B = a + b.$$

Then repeating the n drawings, we obtain

$$P' = \left(\frac{A}{A+B}\right)^n = \left(\frac{A}{a+b}\right)^n$$

as the probability that the balls drawn will be *all white.*

Fermat's theorem may then be stated as follows. *It is impossible so to repaint the balls that for $n>2$, we have*

$$P = P';$$

for this would give

$$a^n + b^n = A^n.$$

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, N. Y.

CLUB REPORTS 1945–46

Pi Mu Epsilon, Northwestern University

There were twelve meetings during the academic year 1945–1946, two of which were held during the summer of 1945. The main event of the year was an open house and banquet sponsored in conjunction with the northwestern Mathematical Colloquium to celebrate the moving of the mathematics department into new quarters. Members of the mathematics departments of the University of Chicago and the Illinois Institute of Technology were invited to a tea. The following papers were presented after the tea and after the banquet:

Index theorems in the calculus of variations, by Professor M. R. Hestenes of the University of Chicago.

Experiences of an editor, by Professor L. R. Ford of the Illinois Institute of Techology.

This year *Illinois Beta* sponsored an undergraduate mathematical competition. Four prizes consisting of books were awarded as follows: First prize: Henry F. Hunter, senior; second prizes: Marjorie L. Smith, senior; James B. Serrin, junior; and Edmond U. Cohler, sophomore.

The place of mathematics in a liberal education, by Professor E. P. Northrop of the University of Chicago, was the title of the talk following the February initiation of twenty-three members. Seven others were initiated in May.

Program meetings for the year were held as follows:

Multisensory aids in the teaching of mathematics, by Professor E. H. C. Hildebrandt.

Finite differences, by Miss D. M. DeWitt.

Transformations, by R. R. Reynolds.

Some mathematical aspects of atomic energy, by Professor H. T. Davis.

Infinite products, by Martinus Esser.

Quality control, by Professor M. E. Wescott.

Greek mathematics in relation to modern mathematics, by Professor E. Hellinger.

Transcendental numbers, by H. F. Hunter.

Introduction to lattic theory, Professor J. W. Givens.

Officers for the year 1945–1946 were: President, H. F. Hunter; Vice-President, Margaret McWhirter; Secretary, Nelda Gulbransen; Treasurer, Helmut Abt. Officers for the year 1946–1947 are: President, Helen Gray; Vice-President, Rose Ann Grundman; Secretary, Marjorie Smith; Treasurer, Arlene Lindahl.

Pi Mu Epsilon, St. Louis University

Lectures were given at the three regular meetings of the Missouri Gamma Chapter of *Pi Mu Epsilon* during 1945–46:

The calculus of variations, by the Rev. Walbert Kalinowski, O.S.B.

The elementary transcendental functions, by Mr. Clarence Wallen, S.J.

The shot effect, by Miss Helen Ann Jackson.

The theory of braids was the title of Professor Emil Artin's talk, which was the Annual Mathematics Lecture, following the April initiation in the Medical School Auditorium. More than 250 members and guests of the Chapter attended.

At the banquet which culminated the day's activities, the Rev. Patrick J. Holloran, S.J., President of St. Louis University, delivered the principal address, welcoming the returning members from the armed services and expressing his personal and the University's interest in the growth of the mathematical activities on the campus.

The officers for 1945–46 were: Director, Miss Helen Ann Jackson; Vice-Director, Herbert Gebhart; Secretary-Treasurer, Mary Nawrocki. Professor Francis Regan has been the Corresponding Secretary and Faculty Adviser.

Kappa Mu Epsilon, Upsala College

Nine meetings were held throughout the year, at four of which the following papers were presented by the members.

Mathematical terminology, by Audrey Richter.

The trisection of an angle, by Arnold Goust.

The psychology of arithmetic, by Natalie Manno.

Mathematics and the imagination, by Marian Larsen.

Cooperative discussions and a trip to the Hayden Planetarium of New York City occupied three meetings. At the midyear special meeting a discussion of probability was presented through

The concept of the random walk, with applications to war problems, by Mr. Edward Molina, formerly research engineer with the Bell Telephone Laboratories.

Our annual banquet and initiation was held on May 31, when three new members were admitted to the chapter.

Mathematics as related to the work on an actuary, given by Mr. Walter Klem of the Mutual Life Insurance Company of New York City, was the evening address.

Officers for the year 1945–1946 were: President, Audrey Richter; Vice-President, Mrs. Mary McKim; Treasurer, Marian Larsen; Secretary, Natalie Manno; Historian, Betty Rudeback; Faculty Sponsor, Professor M. A. Nordgaard.

Pi Mu Epsilon, Michigan State College

For the initiation meeting in May 1945, the program consisted of a series of five minute talks on

Interesting mathematical curves, by the initiates Barbara Brisbin, Portia Treend, David Zaukelies, William Coleman, Patricia Wilson, Barbara Gregory, Martha Driver, Rosalie Manz, Carolyn Trimm, and Joan Bowman.

After an inactive summer term, programs were continued as follows:

Computation of life insurance premiums, by Barbara Brisbin, with computations by Dr. J. S. Frame of net premiums on policies suggested by members of the audience.

Riddles in mathematics, presented from Northrup's book of that title by the initiates Charles Costa, Wesley Porterfield, Marjorie Vis, Clarence Healey, Raymond Hurlburt, William Krupp, Robert Jackson, Katherine Weiss, Mary Fuss, Eleanor Bessonen, Beverly Holcomb, Virginia Snyder, Joanne Attwood, and Doris Rinehart.

Zeller's congruence, by Rosalie Manz, and

Kraitchik's nomogram to solve Zeller's congruence, by Barbara Gregory.

String constructions, by Dr. B. M. Stewart at the fall picnic in the forestry cabin, with all hands strung up in attempted imitation.

Doughnuts, by Dr. B. M. Stewart.

Secret and urgent, presented from Fletcher Pratt's book of that title by the initiates Harlan Ogle, Phyllis Rowe, Floyd Lamb, and Joyce Conklin.

Selected topics from statistics, including

Multinomial theorem applied to smelling salts tests, by Mary Baumgarten.

Interesting curves pertaining to the parabola, by Margaret Fox.

Comparison of the old witch to the normal curve, by Marjorie Vis.

Fitting a straight line to data by least squares, by Jane Walker.

A tribute to Georg Cantor, by Dr. F. Herzog, presented after the winter term banquet.

The arbelos or shoemaker's knife, by Eleanor Bessonen.

The trimetric ruler, by Eunice Anderson.

Coin weighing, by Robert Jackson.

Three Louis C. Plant scholarship awards were presented at the February banquet to outstanding Juniors majoring in mathematics and deemed most worthy on the basis of scholarship, interest in mathematics, and helpfulness to the Department of Mathematics. Barbara Gregory received a first prize of fifty dollars, and Rosalie Manz and Carolyn Trimm split the second prize, receiving fifteen dollars each. A spring picnic was held in the East Lansing State Park.

The following officers were elected for 1946–47: President, Barbara Gregory; Vice-President, Virginia Snyder; Secretary-Treasurer, Charles Costa; Membership Committee, Rosalie Manz, Carolyn Trimm, and Phyllis Rowe.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Analytic Geometry, Third Edition. By F. S. Nowlan. New York, McGraw-Hill Book Co., 1946. 14+355 pages. $2.75.

Here is a rarity indeed: an elementary geometry text which seems to be mathematically sound throughout. The book contains all that anyone would be likely to give in a first course on plane analytic geometry, together with enough of the three-dimensional theory to form a basis for a first course in calculus. It seems unnecessary to go into detail; however we shall point out certain unusual features of the book.

In the plane geometry the emphasis is laid upon direction cosines, though the slope concept is carefully discussed. Equality of two ordered pairs of real numbers $x:y$, $a:b$ is defined as follows: $x:y=a:b$ if and only if $x=ka$, $y=kb$ for some non-zero k. By use of this symbolism the inelegant possibility of zero denominators is entirely avoided. However, the author is not unconscious of the need for the notion of slope in other connections; indeed a later chapter discusses ordinary and partial derivatives of polynomials. These concepts are introduced from a geometrical point of view, with as much rigour as a student is likely to encounter in any first course in calculus.

One would hardly expect to find in these days a real contribution to the theory of polar coordinates; yet the author has treated this subject from a fresh and worthwhile point of view. He conceives of three types of polar coordinates, one in which the radius vector is non-negative, one in which it is non-positive, and finally a "mixed" type. As a result he is able to handle all questions both with rigour and with unusual ease, and his methods, moreover, are particularly useful in the calculus.

The distinction between a necessary and a sufficient condition is pointed out at an early stage in the text and is used throughout. This is particularly obvious in two contrasting chapters: Chapter 3, entitled "Loci and Their Equations" and Chapter 4, of which a partial title is "Equations and Their Loci."

A great deal of care seems to have gone into this book. The reviewer noted a somewhat obscure sentence near the bottom of p. 67, an incomplete statement about a limit at the top of p. 81, and a slight slip in the statement of Corollary 1, p. 107, but could find nothing else amiss. Apparently a textbook need not be incorrect to be elementary!

R. H. BRUCK

Analytic Geometry. By F. D. Murnaghan. New York, Prentice-Hall, Inc., 1946. 8+402 pages. $3.25.

This is the most comprehensive American text on ordinary analytic geometry. Within its covers is much mathematics that is not usually to be found in such texts—much, in fact, that seems to be the author's own development. The method of vectors and matrices is used throughout and the applications of analytic geometry to engineering and physical problems are continually kept in view. The standards of rigor are high, the theoretical explanations are detailed.

The order of presentation of the material is strictly rational. The first chapter is concerned with points and vectors on a line, the second with points, vectors and lines in a plane, the third, fourth and fifth with plane vectors and vectors in space. Chapter five is an excellent treatment of determinants and matrices. It is a pleasure to read in an elementary text that one must "distinguish clearly between the *matrix*, which is a block of four numbers, and the *determinant*, which is a single number."

Now come the conics. Chapter VI is devoted to the circle and the sphere, Chapters VII, VIII and IX to the parabola, ellipse and hyperbola, respectively. The treatment of these familiar curves is not always along conventional lines, however. In the chapter on the circle, for instance, we find the concepts of point and line at infinity and the principle of duality, ideas which are usually not included in Cartesian geometry.

Chapter X is concerned with the quadric surfaces. The method of vectors and matrices has certain definite advantages in this treatment, but it may seem strange to one brought up in the old school to see the equation of a hyperboloid written in the form

$$(v'Av) - 1 = 0.$$

The concluding two chapters are devoted to that difficult subject, the reduction of the general equation of the second degree to canonical forms under translations and rotations by means of their invariants. Such reductions are to be found in the mathematical literature, but this seems to be the first textbook on analytic geometry which gives a complete reduction with characterization of the parameters.

At one point the author's desire for careful statement seems to have overshot the mark. Like several other recent writers, he speaks of *an* equation of a definite locus. This implies that there can be equivalent equations which are not identical, and immediately makes it necessary to explain when two equations are identical. Presumably $2x+3y-5=0$ and $4x+6y-10=0$ are not identical, but what about $2x+3y-5=0$ and $2x+3y=5$, or $3y+2x-5=0$, or $x+x+3y-5=0$? Any definition of identity of two equations would have to be artificial, for the only genuine relation of equality among equations is that of equivalence. It seems proper, therefore, to speak of *the* equation of a given straight line, for there is but one such equation in the sense of equivalence.

Upon reading the author's preface, the reviewer finds some ideas which he wishes to second heartily, and some which raise grave doubts. He has only the desire to echo the statement that "The demands of modern physics and engineering for a knowledge of the elements of calculus are so pressing that it is no longer practicable to postpone the teaching of calculus to the sophomore year." If our American high schools paid proper attention to the education of the gifted pupils, they could easily send them to college with a knowledge of analytic geometry. European schools have done this, and so can we.

It is only when the author states that "the methods used here are not too advanced for college Freshmen, and in fact are suitable for presentation in high school" that the reviewer, with tears and regrets, is forced to disagree. The book before us is too complete, too advanced in its concepts and too sophisticated in its notation for the high school pupil or for the student fresh from high school. Only the genius would survive, but he would then be a mathematician. But the book would seem to be ideal for a second course in analytics.

Since perfect rigor is a will-o'-the-wisp that seems to be able to stay just beyond our grasp, would it not be justifiable in our elementary teaching to admit that there are levels of rigor suitable for students of differing degrees of development? High school geometry is not rigorous, yet it has had an effective influence upon our philosophy. As we progress up the educational ladder, one of the sources of greatest pleasure is in looking down from a higher rung upon the more laborious and less sophisticated methods which we at one time so admired.

"There are also celestial bodies, and bodies terrestrial: but the glory of the celestial is one, and the glory of the terrestrial is another."

C. C. MacDuffee

NEW BOOKS RECEIVED

College Algebra. By A. A. Albert. New York, McGraw-Hill Book Co., 1946. 12+278 pages. $2.75.

College Algebra. By M. W. Keller, Boston, Houghton Mifflin Co., 1946. 471+12 pages. $3.00.

Curves. By R. C. Yates. West Point, U. S. Military Academy, 1946. 230+6 pages.

Differential and Integral Calculus. Second Edition. By R. R. Middlemiss. New York and London, McGraw-Hill Book Co., 1946. 8+497 pages. $3.25.

Mathematics and Life. By F. B. Knight, J. W. Studebaker, and G. Tate. Chicago, Scott Foresman Co., 1946. Book 1: 480 pages, $1.04. Book 2: 512 pages, $1.08.

Trigonometry. Second Edition. By H. K. Hughes and G. T. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd. 7+175+117 pages, 1946. $2.50.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Department of Mathematics, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 741. *Proposed by William Scott, Columbus, Ohio*

Prove that in a (non-degenerate) right spherical triangle with hypotenuse c and legs a, b we have $a^2+b^2>c^2$.

E 742. *Proposed by B. F. Laposky, Cherokee, Iowa*

Let ABC be a triangle, LMN the median triangle, DEF the orthic triangle, O the circumcenter, J the nine-point center, and T, U, V the other intersections of the medians AL, BM, CN with the nine-point circle (J). Now there are two sets of circles tangent to the circumcircle at the vertices A, B, C and also tangent to (J). Show that the circles of one set have their centers at the intersections of OA, OB, OC with the corresponding sides of LMN and touch (J) at D, E, F; the circles of the other set have their centers at the intersections of OA, OB, OC with the lines JT, JU, JV and touch (J) at T, U, V.

E 743. *Proposed by E. P. Starke, Rutgers University*

Determine the conditions on the constants a and b such that

$$\log x - ax + b = 0$$

shall have two real solutions, one real solution, or no real solutions.

E 744. *Proposed by Paul Erdös, Stanford University*

Let $a_1<a_2<\cdots<a_n\leqq 2n$ be n positive integers such that the least common multiple of any two is greater than $2n$. Then $a_1>[2n/3]$.

E 745. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If, in an arithmetic progression of n terms and common difference d, n is prime to each term, then n is not prime to d.

SOLUTIONS

Exponents Yielding Only Odd Binomial Coefficients

E 706 [1946, 97]. *Proposed by D. H. Browne, Buffalo, N. Y.*

For what values of n does $(a+b)^n$ yield only odd coefficients?

I. *Solution by E. P. Starke, Rutgers University.* In order that all coefficients be odd, each ratio of consecutive coefficients,

$$\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \cdots, \frac{n-k}{k+1}, \cdots, \frac{1}{n}, \tag{1}$$

must reduce to a fraction in simplest terms with both numerator and denominator odd. Evidently n itself is odd. Let $n=2^s u-1$, where u is odd. Then the ratio of the (2^s+1)th term to the 2^sth term is

$$\frac{n-(2^s-1)}{2^s} = \frac{2^s(u-1)}{2^s} = \frac{u-1}{1},$$

which reduces to a form having an even numerator over an odd denominator. This is contrary to our requirements unless $u=1$ so that $n=2^s-1$ and there is no (2^s+1)th term. Conversely, if $n=2^s-1$, the general term of the sequence (1) is $[2^s-(k+1)]/(k+1)$ with $k<n$. If $k+1=2^t v$, v odd, this ratio becomes $(2^{s-t}-v)/v$, which meets our requirements. Thus n is a power of 2 diminished by unity.

II. *Solution by Paul Bateman, Philadelphia, Pa.* We claim that if the binary expansion of n contains no zeros (*i.e.*, if n is of the form 2^k-1), then all the binomial coefficients are odd, and that if, on the contrary, the first zero counting from the right in the binary expansion of n is in the rth place, then the binomial coefficient $\binom{n}{2^{r-1}}$ is even (and is, in fact, the first such).

To this end we note that $\binom{n}{j+1}$ is obtained from $\binom{n}{j}$ by multiplying by $(n-j)/(j+1)$, $j=0, 1, \cdots, n-1$. If the first zero counting from the right in the binary expansion of j is in the r_jth place, then $j+1$ has a 1 in the r_jth place and zeros to the right thereof. Thus $j+1$ has 2 as a factor r_j-1 times. If the last r_j digits (at least) in the binary expansion of n are 1's, then $n-j$ has a 1 in the r_jth place an all zeros to the right thereof, and consequently has 2 as a factor r_j-1 times. Thus in going from $\binom{n}{j}$ to $\binom{n}{j+1}$ we do not pick up a factor 2. On the other hand, if the binary expansion of n has a zero in the rth place and only 1's to the right thereof, and if j is that number whose binary expansion consists of just $r-1$ digits 1, then $n-j$ will have 2 as a factor at least r times, whereas $j+1=2^{r-1}$. Thus in going from $\binom{n}{j}$ to $\binom{n}{j+1}$ we do pick up (for the first time) a factor 2.

III. *Solution by M. S. Knebelman, Washington State College.* The answer to the question stated is that n must be of the form 2^s-1. Since this is a special case of a more general problem that leads to an interesting conclusion, we restate the problem: What must be the form of n so that for all k, C_k^n is relatively prime to a prime $p<n$?

We shall say that a set of number $\{N_k\}$ is relatively prime to p if each member of the set is relatively prime to p. Let α be the greatest exponent such that $p^\alpha \leqq n+1$. Then since $kC_k^n = C_{k-1}^n(n+1-k)$ and $\{C_k^n\}$ is relatively prime to p, if we let $k=p^\alpha$ it follows that $n+1$ must be a multiple of p^α. Hence $n=ap^\alpha-1$, where $a<p$. This condition is necessary; that it is sufficient can be observed from the fact that $C_k^n = C_{k-1}^n(ap^\alpha-k)/k$. For in this case C_k^n cannot contain p as a factor if $k<p$, and if k contains p as a factor, $(ap^\alpha-k)/k$ does not, since $a<p$. If we require that $\{C_k^n\}$ be also relatively prime to another prime q, then $n=ap^\alpha-1=bq^\beta-1$, $b<q$. Hence $ap^\alpha=bq^\beta$. Suppose $q>p$. Then, from the last equation, we must have $b=cp^\alpha$, so that $a=cq^\beta>p$, which is a contradiction. Hence the set $\{C_k^n\}$ can be relatively prime to at most one prime p. We can combine these results into one theorem:

Let $n+1=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, *where* $p_1<p_2<\cdots<p_r$. *Then if* $p_1^{\alpha_1}\cdots p_{r-1}^{\alpha_{r-1}} < p_r$, $\{C_k^n\}$ *is relatively prime to* p_r *and is not relatively prime to any other prime less than* n. *If* $p_1^{\alpha_1}\cdots p_{r-1}^{\alpha_{r-1}} > p_r$, *the set* $\{C_k^n\}$ *is not relatively prime to any prime less than* n.

Also solved by D. W. Alling, Murray Barbour, R. C. Buck, H. S. Grant, H. D. Grossman, A. E. Karp, Nicholas Kushta, E. M. Michalup, P. A. Piza, Irene Price, W. J. Robinson, E. D. Schell, C. W. Topp, J. H. White, B. F. Yanney, and the proposer. Several solvers made use of Pascal's arithmetic triangle.

The Inverse of a Matrix

E 710 [1946, 97]. *Proposed by D. H. Lehmer, Aberdeen Proving Ground, Md.*

Find the inverse of the symmetric matrix of the nth order: $A=\|a_{ij}\|$, in which $a_{ij}=i/j$ for $i\leqq j$.

I. *Solution by D. M. Smiley and M. F. Smiley, Bethlehem, Pa.* Let A_n denote the matrix described and let $\|r_{ij}(n)\|$ denote its inverse (if, indeed, it has one). Direct computation for $n=2, 3$ leads one to suspect that $r_{ij}(n)=0$ if $i\geqq j+2$, and that $r_{ij}(n)=r_{ij}(n-1)$ if $i, j=1, \cdots, n-1$, and $(i, j)\neq(n-1, n-1)$. To test this, introduce numbers a, b, c, and set

$$C_n = \begin{Vmatrix} (A_{n-1})^{-1}+aB_{n-1} & b(T_{n-1})' \\ bT_{n-1} & c \end{Vmatrix},$$

in which T_{n-1} is the row matrix $(0, \cdots, 0, 1)$ of order $n-1$ and B_{n-1} is the square matrix $(0, \cdots, 0, (T_{n-1})')$ of order $n-1$. In order that $A_nC_n=I$, we must have $c=n^2/(2n-1)$, $b=-n(n-1)/(2n-1)$, and $a=(n-1)^2/(2n-1)$. Conversely, these values yield $A_nC_n=I$. From this result we easily find that

$$r_{ii} = 4i^2/(4i^2 - 1), \qquad (i = 1, \cdots, n-1),$$
$$r_{nn} = n^2/(2n - 1),$$
$$r_{i\,i+1} = -\,i(i+1)/(2i+1), \qquad (i = 1, \cdots, n-1),$$
$$r_{ij} = 0 \quad \text{for} \quad j > i+1.$$

II. *Solution by John Williamson, Queens College.* Let $A=(a_{ij})$ be a symmetric matrix of order n where $a_{ij}=a_i/a_j$, $i\leqq j$, $a_i\neq 0$ and $a_i\neq a_j$ if $i\neq j$. If

$$B = \begin{Vmatrix} 1 & -b_1 & 0 \cdots 0 & 0 \\ 0 & 1 & -b_2 \cdots 0 & 0 \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ 0 & 0 & 0 \cdots 1 & -b_{n-1} \\ 0 & 0 & 0 \cdots 0 & 1 \end{Vmatrix},$$

where $b_i=a_i/a_{i+1}$, $i=1, 2, \cdots, n-1$, $BAB'=D$, where $D=[d_1, d_2, \cdots, d_n]$ is a diagonal matrix $d_i=1-b_i^2$, $b_n=0$. Further, $(B')^{-1}A^{-1}B^{-1}=D^{-1}$, so that $A^{-1}=B'D^{-1}B$. Since D^{-1} is a diagonal matrix $[k_1, k_2, \cdots, k_n]$, $k_i=1/d_i$, a simple matrix calculation gives $A^{-1}=R=(r_{ij})$, where

$$r_{ii} = k_i + b_{i-1}^2 k_{i-1}, \qquad b_0 = 0,$$
$$r_{i\,i+1} = r_{i+1\,i} = -\,b_i k_i,$$
$$r_{ij} = 0, \qquad |\,i-j\,| \geqq 2.$$

On substituting for k_i and b_i their values we find

$$r_{ii} = a_i^2(a_{i+1}^2 - a_{i-1}^2)/(a_{i+1}^2 - a_i^2)(a_i^2 - a_{i-1}^2), \qquad i = 1, 2, \cdots, n-1,$$
$$r_{nn} = a_n^2/(a_n^2 - a_{n-1}^2),$$
$$r_{i\,i+1} = -\,a_{i+1}a_i/(a_{i+1}^2 - a_i^2),$$

and if $a_i=i$,

$$r_{ii} = 4i^2/(4i^2 - 1),$$
$$r_{nn} = n^2/(2n - 1),$$
$$r_{i\,i+1} = -\,i(i+1)/(2i+1),$$

$i=1, 2, \cdots, n-1$.

Also solved by D. W. Alling, Paul Bateman, E. A. Cameron, H. S. Grant, and H. P. Hitchcock.

Cyclic Numbers in Geometric Progression

E 714 [1946, 157]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration of base B, find the sets of cyclic numbers of three

different digits such that the numbers of each set are successive terms of a geometric progression.

If $B=3p+1$ show that the three digits of at least two of these sets are consecutive terms of an arithmetic progression. Apply to the case $B=10$.

Solution by E. P. Starke, Rutgers University. The present problem is a generalization of E 658 [1945, 461] such that the earlier problem provides the solution in case $B=10$. As in that solution, let us put

$$x = B^2a + Bb + c, \qquad y = B^2b + Bc + a, \qquad z = B^2c + Ba + b.$$

If y is the geometric mean, we have $y^2=xz$, which is easily reduced to

$$B = (a^2 - bc)/(b^2 - ac), \tag{1}$$

or

$$c = (B^3 - 1)a^2/(Ba - b) - Bb - B^2a. \tag{2}$$

(In the solution of E 658 it was erroneously assumed that a common divisor of a, b is necessarily also a divisor of c. Even without this assumption, however, no further results are found.)

For a specified value of B, solutions may be found by trial: assign a value to a, determine values of $b(\neq a)$ for which the fraction in (2) is an integer; a solution results if $0 \leqq a, b, c < B$. For example if $B=11$, (2) becomes

$$c = 1330a^2/(11a - b) - 11b - 121a.$$

Let $a=1$; then $11-b$ divides 1330 for $b=1$, 4, 6, 9. Disregarding $b=a=1$, all other values of b give $c>11$. Let $a=2$; then $22-b$ divides 5320 for $b=2$, 3, 8. Here $b=3$ yields $c=5$. Continuing thus we find all the solutions: $(a, b, c) = (2, 3, 5)$, $(4, 6, X)$, $(6, 3, 1)$, $(9, 4, 1)$, where X is the digit ten. This procedure is tedious for large values of B, but no more efficient method appears.

If $B=6$, 8, 12 or any value for which $B-1$ and B^2+B+1 are both prime, it is easy to see that there are no solutions.

The second part of the problem is shown immediately upon observing that

$$\begin{aligned} &B = 3p + 1, \quad a = p + 1, \quad b = p, \quad c = p - 1; \\ &B = 3p + 1, \quad a = 2p + 2, \quad b = 2p, \quad c = 2p - 2; \end{aligned} \tag{3}$$

reduce (1) to an identity. (Other, less simple, identical solutions are easy to find: *e.g.* $b=2c-1$, $a=4c-4$, $B=14c^2-31c+16$.)

Conversely, suppose a, b, c are consecutive terms of an arithmetic progression, so that $a=2b-c$. Then (1) becomes

$$B = (4b - c)/(b - c) = 1 + 3b/(b - c).$$

Now $b-c<0$, $b-c\geqq 3$ lead to contradiction of the requirement $0 \leqq a, b, c < B$. If $b-c=1$, 2, we are led at once to (3). There are never more than two sets.

Also solved by Murray Barbour, E. D. Schell, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, N. J. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4220. *Proposed by P. Erdös, University of Michigan*

Let m be a positive integer with no prime factors greater than the integer n and $m \leqq n^{(k+1)/2}$. Then m can be written as the product of k integers $\leqq n$; the exponent $(k+1)/2$ is the best possible.

4221. *Proposed by P. Erdös, University of Michigan*

Let $\phi(n) = n\Pi_{p/n}(1-1/p)$ be Euler's function. Prove that for every k the equation $\phi(x) = k!$ is solvable.

4222. *Proposed by J. H. Butchart, Grinnell College*

If points are numbered and if $\overline{12}$ denotes the distance from 1 to 2, then

$$\begin{vmatrix} 0 & \overline{12}^2 & 1 \\ \overline{21}^2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2L^2,$$

where L is the length of the segment. Generalize this for the triangle and tetrahedron showing that the corresponding determinants are $-16A^2$ and $288V^2$ respectively, where A is the area of the triangle and V is the volume of the tetrahedron.

4223. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron the harmonic plane of the point L whose normal coordinates are proportional to the radii of the circumcircles of the triangles of the faces (second Lemoine point), coincides with the polar plane with respect to the circumsphere.

Dedicated to N. A. Court.

4224. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$; (1) The right cones with vertices at the orthogonal projections of the second Lemoine point L on the axes of the circumcircles of the faces and with these circles as bases have the same base angle V (Brocard angle). (2) The symmedians AL, BL, CL, DL meet the circumsphere in the ver-

tices of the tetrahedron $A'B'C'D'$ having the same Brocard angle V and the same Lemoine point L as $ABCD$.

Dedicated to N. A. Court.

SOLUTIONS

A Differential Operator

4156 [1945, 220]. *Proposed by Charles M. Stein, New York City*

Show that if $f(u)$ is a polynomial in u and $D=d/dx$, then for any positive integer n

$$\left\{1+\frac{[(k-1)n]!}{[kn]!}(y^n-x^n)D^n\right\}\cdots\left\{1+\frac{1}{n!}(y^n-x^n)D^n\right\}f(x^n)=f(y^n).$$

If possible, generalize the above result to analytic functions $f(u)$ as in the special case $n=1$ in 3985 [1942, 345].

Solution of first part by the Proposer. We have

$$\left\{1+\frac{[(s-1)n]!}{[sn]!}(y^n-x^n)D^n\right\}x^{ns}=x^{ns}+(y^n-x^n)x^{(s-1)n}=y^nx^{(s-1)n}.$$

Consequently

$$\left\{1+\frac{(y^n-x^n)}{n!}D^n\right\}\cdots\left\{1+\frac{[(s-1)n]!}{[ns]!}(y^n-x^n)D^n\right\}x^{ns}=y^{ns}.$$

But the factors in the operational product are commutative, since any linear operator, and in particular $(y^n-x^n)D^n$, is commutative with powers of itself. Thus

$$\cdots\left\{1+\frac{[(k-1)n]!}{[kn]!}(y^n-x^n)D^n\right\}\cdots\left\{1+\frac{(y^n-x^n)}{n!}D^n\right\}x^{ns}$$
$$=\cdots\left\{1+\frac{[sn]!}{[(s+1)n]!}(y^n-x^n)D^n\right\}y^{ns}=y^{ns}.$$

This shows that the theorem is true for any finite linear combination of powers of x^n, *i.e.*, any polynomial in x^n.

Conics

4162 [1945, 345.] *Proposed by H. S. M. Coxeter, University of Toronto*

Prove, by the methods of real projectivity geometry, that if a projectivity $P\overline{\wedge}P'$ on a conic is not an involution, the envelope of PP' is a conic. (For a complex proof, see H. F. Baker, *Principles of Geometry*, Cambridge 1922 or 1930, p. 52.)

Solution by R. J. Walker, Cornell University. It is convenient to break the proof into parts.

LEMMA. *Given a conic* Γ *and two lines* a *and* b *not conjugate with respect to* Γ *and not intersecting on* Γ, *there is a conic* Δ *tangent to* a *and such that any point of* b *has the same polar with respect to both* Γ *and* Δ.

Proof. Let $P=ab$, and let Q be any other point of b not on Γ and not conjugate to P with respect to Γ. Let p and q be the polars of P and Q with respect to Γ and let $A=ap$. Let AQ meet q in X and let A_1 be the harmonic conjugate of A with respect to Q and X. Let A_1P intersect p in Y and let A_2 be the harmonic conjugate of A_1 with respect to P and Y. Let A_1P intersect q in Z and let A_3 be the intersection of AZ and A_2Q. It is easily verified that no three of A, A_1, A_2, A_3 are collinear and that none of A_1, A_2, A_3 lie on a. Hence there is a conic Δ containing A, A_1, A_2, A_3 and tangent to a at A. It is also readily verified that p and q are the polars of P and Q with respect to Δ. Now let R be any point of b. The point $B=pq$ is the pole of b with respect to both Γ and Δ, and so the polars of R with respect to Γ and Δ both pass through B. Γ and Δ determine the same involution of conjugate points on b, since the two pairs P, bp and Q, bq are conjugate with respect to both conics. Hence there is a point R' on b conjugate to R with respect to both conics. The line BR' is thus the polar of R with respect to both Γ and Δ.

THEOREM. *Let* A, A' *be a pair of distinct points in a projectivity (not an involution) on a conic* Γ, *and let* b *be the axis of the projectivity. Then the conditions of the Lemma are satisfied with* $a=AA'$, *and every line* PP' *joining a pair of points of the projectivity is tangent to* Δ.

Proof. Let PA' and $P'A$ meet at Q on b. Let S be the intersection of AA' and PP', and R of AP and $A'P'$. Then SR is the polar of Q with respect to Γ and hence also with respect to Δ. Therefore SR and SQ are harmonic with SA and the other tangent to Δ from S. But from the quadrangle $AA'PP'$, SQ and SR are harmonic with SA and SP. Hence $SP(=PP')$ is tangent to Δ.

Solved also by Howard Eves and James Jenkins.

Cubes with Twelve Digits

4164. [1945, 346]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find a number of twelve digits $a_1a_2a_3a_4b_1b_2b_3b_4c_1c_2c_3c_4$ which is a perfect cube, and the cube root is the sum of the numbers $a_1a_2a_3a_4$, $b_1b_2b_3b_4$, $c_1c_2c_3c_4$.

Solution by E. P. Starke, Rutgers University. Let $A=a_1a_2a_3a_4$, $B=b_1b_2b_3b_4$, $C=c_1c_2c_3c_4$. Then

$$(1) \qquad (A+B+C)^3 = 10^8A + 10^4B + C \equiv A + B + C \pmod{9999}.$$

This congruence is equivalent to the three similar simultaneous congruences having the moduli 9, 11, 101. These are satisfied respectively if and only if

$$A + B + C \equiv 0, 1, -1 \pmod 9,$$
$$A + B + C \equiv 0, 1, -1 \pmod{11},$$
$$A + B + C \equiv 0, 1, -1 \pmod{101}.$$

Thus the 27 solutions of congruence (1) are easily obtained. They are 1, 100, 505, 2222, 2223, 2322, 2727, 2728, 2827, 4445, 4544, 4949, 4950, 5049, 5050, 5455, 5554, 7172, 7271, 7272, 7677, 7776, 7777, 9494, 9899, 9998, 9999.

Since $(A+B+C)^3$ must be a 12-digit number, we have also

$$4641 < A + B + C < 10000.$$

Upon finding the cubes of the above numbers we have the results:

$$4949^3 = 121213882349; \qquad 1212 + 1388 + 2349 = 4949.$$
$$5049^3 = 128711132649; \qquad 1287 + 1113 + 2649 = 5049.$$
$$5455^3 = 162324571375; \qquad 1623 + 2457 + 1375 = 5455.$$
$$5554^3 = 171323771464; \qquad 1713 + 2377 + 1464 = 5554.$$
$$7172^3 = 368910352448; \qquad 3689 + 1035 + 2448 = 7172.$$

1, 2322, 2728, 4445, 4544 also satisfy the requirements except for the necessity of supplying one or more zeros as initial digits of the cube. All other numbers cited above are such that

$$(A + B + C)^3 = 10^8A + 10^4B + C - 9999.$$

Solved also by Hugo Brandt and the proposer without stating methods of derivation, Brandt giving one example with the remark "Here is one needle in the haystack." The proposer gave three.

Circles Intersecting on a Parabola

4166. [1945, 400]. *Proposed by J. H. Butchart, Grinnell College*

Given the straight lines l, l' and a point F. A variable circle through F and the intersection of l, l' cuts l and l' in A and A' respectively. The circles through F and tangent to l, l' at A, A' respectively meet again on a parabola tangent to l and l' and having F as focus.

I. *Solution by H. F. Sandham, Trinity College.* The circumcircles of the triangles formed by a complete quadrilateral intersect in the focus of the unique parabola touching its sides. This is a limiting case when two of the sides coincide. Take two of the sides to be l, l' and let the other two intersect in P and tend to coincidence with l''. Then the circles passing through P and touching l, l' at their points of intersection with l'', *i.e.* A, A' respectively, and the circumcircle of l, l', l'' intersect in the focus F of the unique parabola touching l, l', and l'' at P.

II. *Solution by Howard Eves, College of Puget Sound.* As a special case of the first theorem in Butchart's article *Some Properties of the Limaçon and Cardioid* (this MONTHLY [1945, 384]), we have the theorem:

Given two circles l and l' intersecting in points F and E. Through E we draw an arbitrary secant cutting the circles l and l' again in A and A' respectively.

The tangents to l and l' at A and A' meet in a point P on a cardioid tangent to l and l' and having its cusp at F.

Inversion of this theorem with respect to F as center of inversion produces the required result.

Solved also by Mrs. Leslie Davis and P. D. Thomas.

Editorial Note. The solution by Mrs. Davis is analytic, and Thomas gave two solutions, one analytic and the other synthetic. Another solution follows.

The tangents at two points P_1 and P_2 on a parabola meet in the point C, then it is known that the tangent segments CP_1 and CP_2 subtend equal angles at the focus F; that in the quadrilateral P_1FP_2C the points P_1, P_2 are on opposite sides of CF; and the equal angles at F are equal to the exterior angle at C of the quadrilateral.

Now let the tangent at a third point P_3 meet the straight line of CP_2 in A and CP_1 in B, forming the triangle ABC with the interior angles α, γ at A, C, respectively, and the exterior angle β at B. Then by the theorem above P_3A and AP_2 subtend the angle α at F; P_2C and CP_1 subtend the angle γ at F; and P_3B and BP_1 subtend the angle β at F. Thus at F there are three pairs of equal angles such that $2\alpha+2\gamma=2\beta$; and hence $\angle AFC=\beta$, which says that the circumcircle of ABC passes through F. Obviously the circle through F and tangent to CA at A passes through P_3, and similarly the circle through F and tangent to CB at B goes through P_3, *etc.*

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

In memory of Professor Federigo Enriques a group of his students have established a "Grant Federigo Enriques." The object of the fund is to assist some of his pupils to continue their scientific careers. Those interested in assisting in this enterprise are asked to write to Casa Editrice Zanichelli, Bologna, Italy.

Ohio State University announces the following: Assistant Professor Grace M. Bareis has retired, Assistant Professor Marshall Hall of Yale University has been appointed to an associate professorship, and Associate Professor H. H. Alden of the University of Wyoming has been appointed to an assistant professorship.

Dr. Joshua Barlaz of Ohio State University has been appointed to an assistant professorship at Rutgers University.

B. K. Brown of Syracuse University has been appointed to an associate professorship at the James Millikin University.

Dr. R. S. Burington, Director of the evaluation and analysis groups of the research and development division of the bureau of ordnance, Navy Department, has been appointed chief mathematician, bureau of ordnance.

P. C. Cox has been appointed to an assistant professorship at Roanoke College.

Assistant Professor Howard Eves of the College of Puget Sound has been appointed to an assistant professorship at Oregon State College.

Assistant Professor W. J. Harrington has received a leave of absence from Pennsylvania State College to accept a visiting assistant professorship at Cornell University.

J. F. Hubbard has been appointed to an assistant professorship at Massachusetts State College at Fort Devens.

Associate Professor H. S. Kaltenborn of the University of Idaho has been appointed professor of mathematics and head of the department at Memphis State College.

Associate Professor V. S. Lawrence of Cornell University has been appointed to an associate professorship at Virginia Polytechnic Institute.

Professor Deane Montgomery of Smith College has been appointed to an associate professorship at Yale University.

Dr. A. R. Poole has been appointed to an assistant professorship at Oregon State College.

Assistant Professor R. G. Sanger of the University of Chicago has been appointed professor of mathematics and head of the department at Kansas State College. Professor W. T. Stratton has retired from his position as head of the department.

Dr. R. L. Westhafer has been appointed to an assistant professorship at New Mexico College of Agricultural and Mechanic Arts.

The following appointments to instructorships are announced:
Clark Junior College, Vancouver, Washington: R. H. Stair
Duke University (department of Physics): Rachael La Roe
Lafayette College: Dr. J. C. Smith

Professor Richard Baldus of the Technische Hochschule of Munich died in January, 1945.

Professor Friedrich Hartogs, formerly of the University of Munich, died August 15, 1943.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE WATSON SCIENTIFIC COMPUTING LABORATORY

In the early part of 1945 the International Business Machines Corporation created a Department of Pure Science under the direction of Dr. W. J. Eckert. One of the first projects proposed by the new department was the establishment of a computing laboratory to carry on research involving complicated computational techniques and to give instruction in modern numerical methods. Columbia University invited the International Business Machines Corporation to locate the proposed laboratory upon the Columbia University campus, and to occupy a building at 612 West 116th Street; this invitation was accepted. The new laboratory was named The Watson Scientific Computing Laboratory. For Dr. Eckert, who became director of the new laboratory, the affiliation with Columbia University represented a continuation of a relation started several years ago. In 1933, he developed the first computing laboratory in which general scientific calculations were automatically performed. This laboratory was made available to astronomers through The Thomas J. Watson Astronomical Computing Bureau, a joint enterprise of the American Astronomical Society, Columbia University and the International Business Machines Corporation. Some of the rapid computing methods developed there were perfected at the United States Naval Observatory during the period of the recent World War when Dr. Eckert was Director of the Nautical Almanac. All members of the staff of the Watson Laboratory are employees of the International Business Machines Corporation, and some have been appointed to the teaching staff of the University.

The purpose of the Watson Laboratory is research and instruction in the various branches of science, especially those involving applied mathematics and numerical calculation. A free computing service is to be offered to a worker in any field of science if he has a problem that is interesting to the members of the Laboratory staff. Members of the staff offer courses of instruction in their fields of interest under the auspices of various departments of the University; this instruction is designed for graduate students. Academic credit for the courses may be obtained by registering with the University in the usual manner and may be counted toward a University degree when approved by the department and the faculty concerned. An important function of the Laboratory is to serve as a center for the exchange of information concerning mathematical and machine methods and mathematical tables.

The Laboratory is equipped with the most modern calculating machines. These machines are of a wide variety to permit not only the most efficient

handling of a given problem but the comparative study of different methods. Included are standard keyboard and electric punched-card machines as well as special calculators which operate by means of relay networks and electronic circuits. The following are the principal units:

Keyboard machines: calculators, printing-adding machines.

Standard electric punched-card machines: 2 alphabetic accounting machines, 6 multiplying punches, 2 collators, 3 reproducing summary punches, sorters, key punches. These units are equipped with many special devices.

Relay calculators: (*a*) Two automatic calculators of large capacity. These machines have automatic plugboard sequence control, 30 counters of 6- to 12-digit capacity and facilities for consulting mathematical tables. They have provision for division and square root and each is capable of performing and recording 16000 6×6 multiplications per hour. (*b*) Experimental multiplying and dividing machine for sequence calculation. This is a flexible unit which can be attached to other machines for special purposes. (*c*) A small digital calculator for special problems. Its operation is controlled by means of a program card. It operates in the decimal, duodecimal, binary, and other systems.

Electronic calculator: an all-electronic multiplying-dividing machine designed for high speed. Results are recorded on punched cards.

Table printing device: this device reads numbers from punched cards and records them for printing by means of lithography or line cuts. It combines the reliability of punched-card operation with the flexibility and elegance of conventional printing.

Card-controlled measuring and recording machine: an automatic machine designed primarily for the measurement of astronomical photographs. It locates the star image by means of coordinates on a punched card, measures its position, and records the measurement on the card. The punched card record is then available for mathematical treatment.

Continuous calculator: an electronic device for the solution of simultaneous linear equations up to and including the twelfth order.

Facilities for the construction of new equipment required by the Laboratory are available in the Department of Engineering of IBM and in the Laboratory itself.

A technical library is maintained in the Laboratory. It will be highly specialized in the field of activity of the Laboratory because of the proximity of the extensive libraries of the University and of the American Mathematical Society.

NEW PRELIMINARY ACTUARIAL EXAMINATIONS

On June 7, 1947, three new preliminary actuarial examinations will be given to undergraduate students of mathematics and others who may be interested in going into the actuarial profession. These new examinations are sponsored jointly by the Actuarial Society of America and the American Institute of Actuaries.

The new series of examinations will replace Parts 1, 2, and 3 of the actuarial examinations which have been given heretofore and will carry the same credit

toward Associateship in the two actuarial organizations. These examinations have been prepared under the direction of a joint committee of actuaries and mathematicians. They will be administered by the College Entrance Examination Board at centers through the United States and Canada.

Descriptions of the three new examinations are as follows:

1. *Language Aptitude Examination.* This is a three-hour aptitude examination testing reading comprehension and precise knowledge of the meaning of words. It is similar to the well-known Scholastic Aptitude Test of the College Entrance Examination Board, except that it is pitched at approximately the college sophomore level. Verbal facility and command of the English language, as well as mathematical ability, are important in the actuarial profession. This is not the type of examination for which specific preparation can be made; it is an aptitude rather than an achievement examination.

2. *General Mathematics Examination.* This is a three-hour achievement examination on material usually covered in the first two years of mathematics in colleges and universities in the United States and Canada. More specifically, it is based on college algebra, trigonometry, coordinate geometry, and differential and integral calculus. It is designed to be taken by the mathematically talented undergraduate at the end of his sophomore year, although it is not restricted to this group.

3. *Special Mathematics Examination.* This is a three-hour achievement examination based on the material usually covered in undergraduate courses in finite differences, probability, and statistics. It is designed to be given at the end of the junior or senior year to college mathematics majors and others who have either taken courses or done concentrated reading in these fields, but it is not restricted to this group.

The two actuarial bodies will jointly award one $200 and eight $100 prizes to the nine undergraduates ranking highest in combined score on the first two of the examinations described above. In determining these awards the General Mathematics Examination will be weighted twice as much as the Language Aptitude Examination.

Information regarding these new examinations, and applications for taking them, may be obtained from either of the following organizations:

The Actuarial Society of America
393 Seventh Avenue
New York 1, New York

American Institute of Actuaries
135 South La Salle Street
Chicago 3, Illinois

REVISED SELECTIVE SERVICE MEMORANDUM NO. 115

The revised Selective Service Local Board Memorandum No. 115, as released to the public on August 23, 1946, contained the following provisions of interest to mathematicians:

Pursuant to the provision of Part II, paragraph I of this memorandum,* local boards will give the most serious consideration to the occupational deferment of registrants engaged in the specialized, technical and scientific fields set forth in Part IV of this memorandum.

PART IV—STANDARDS AND PROCEDURES

1. *Advanced studies in the physical sciences and engineering.* Registrants accepted by an accredited college or university for a Master's or Doctor's degree in the physical sciences or engineering will be certified by the Office of Scientific Research and Development to the Director of Selective Service.

2. *University research in the physical sciences or engineering.* Registrants employed by or attached to the staff of an accredited college or university for research in the physical sciences or engineering will be certified by the Office of Scientific Research and Development to the Director of Selective Service.

3. *Submission of information in certain cases.* Any registrant who wishes to be certified under the provisions of paragraphs 1 and 2 above must present to the Office of Scientific Research and Development, 1530 P Street, N.W., Washington 25, D.C., the following documents in triplicate:

(a) A notarized statement of his intention:
  (1) To undertake graduate studies leading to a Master's or Doctor's degree in the physical sciences or engineering; or
  (2) To undertake advanced research in the physical sciences or engineering.

(b) A statement from an accredited college or university signed by a responsible official of the college or university, indicating that the registrant:
  (1) Has been accepted as a candidate for a Master's or Doctor's degree in the physical sciences or engineering; or
  (2) Is to be employed by or attached to the staff of the college or university for research in the physical sciences or engineering.

(c) A statement showing the registrant's name, address, age, Selective Service local board number and address, classification, and educational qualifications.

4. *Industrial and foundational research in the physical sciences or engineering.* Scientific personnel necessary to and engaged in basic research in the physical sciences or engineering in the employ of laboratories not connected with colleges or universities, and scientific personnel in industry who are not engaged directly in production, will be certified by the Office of Scientific Research and Development to the Director of Selective Service. Employers will submit Form

* A registrant age 19 through 29 may be retained or placed in Class II-A if the local board finds that he is irreplaceable in and indispensable to an activity essential to the national existence. An activity essential to the national existence is one of such importance that any disruption thereof would adversely affect the physical well-being, the public safety or economic life of the community or the Nation.

42A (Special-Revised) in triplicate, in each individual case, to the Office of Scientific Research and Development for certification.

5. *Teachers.* Teachers employed by an accredited college or university will be certified by the United States Office of Education to the Director of Selective Service. Employers will submit Form 42A (Special-Revised) in triplicate, in each individual case, to the United States Office of Education, Temporary Building "M," 26th and Water Streets, N.W., Washington 25, D. C.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTY-EIGHTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-eighth summer meeting of the Mathematical Association of America was held at Cornell University, Ithaca, New York, on Monday and Tuesday, August 19–20, 1946, in conjunction with the summer meeting and colloquium of the American Mathematical Society and the meeting of the Institute of Mathematical Statistics. Five hundred and eighty-four persons were in attendance at the meetings, including the following two hundred and seventy-seven members of the Association:

C. R. ADAMS, Brown University
R. P. AGNEW, Cornell University
A. A. ALBERT, University of Chicago
E. B. ALLEN, Rensselaer Polytechnic Institute
C. B. ALLENDOERFER, Haverford College
R. C. ARCHIBALD, Brown University
SILVIO AURORA, Columbia University
W. L. AYRES, Purdue University

D. H. BALLOU, Middlebury College
P. T. BATEMAN, Yale University
GRACE E. BATES, Mount Holyoke College
HELEN P. BEARD, Newcomb College
F. P. BEER, Williams College
R. L. BEINERT, Cornell University
J. A. BENNER, Lafayette College
DOROTHY L. BERNSTEIN, University of Rochester
FELIX BERNSTEIN, New York University
GARRETT BIRKHOFF, Harvard University
C. J. BLACKALL, North Baltimore, Ohio
E. E. BLANCHE, War Department
R. P. BOAS, JR., Brown University
S. G. BOURNE, Johns Hopkins University
JULIA W. BOWER, Connecticut College
C. C. BRAMBLE, U. S. Naval Academy
R. W. BRINK, University of Minnesota
B. K. BROWN, James Millikin University
H. E. BUCHANAN, Tulane University
R. C. BUCK, Harvard University
E. L. BUELL, Northwestern University
R. S. BURINGTON, Bureau of Ordnance, Navy Department
SISTER LEONARDA BURKE, Regis College, Massachusetts
F. J. H. BURKETT, Union College
HERBERT BUSEMANN, Smith College
W. H. BUSSEY, University of Minnesota

S. S. CAIRNS, Queens College
W. D. CAIRNS, Oberlin College
R. H. CAMERON, University of Minnesota
C. C. CAMP, University of Nebraska
H. H. CAMPAIGNE, U. S. Navy
W. B. CAMPBELL, Philadelphia Textile Institute
I. S. CARROLL, Syracuse University
W. B. CARVER, Cornell University
W. F. CHENEY, JR., University of Connecticut
J. A. CLARKSON, University of Pennsylvania
MARY D. CLEMENT, Wells College
G. R. CLEMENTS, U. S. Naval Academy
NANCY COLE, Connecticut College

R. H. COLE, University of Western Ontario
E. P. COLEMAN, U. S. Military Academy
J. B. COLEMAN, Columbia, S. C.
J. A. COOLEY, University of Tennessee
BYRON COSBY, JR., U. S. Naval Academy
N. A. COURT, University of Oklahoma
W. H. H. COWLES, Pratt Institute
H. S. M. COXETER, University of Toronto
H. B. CURRY, Pennsylvania State College
J. H. CURTISS, National Bureau of Standards

D. R. DAVIS, State Teachers College, Montclair, N. J.
F. F. DECKER, Syracuse University
L. S. DEDERICK, Aberdeen Proving Ground
H. L. DORWART, Washington and Jefferson College
ARNOLD DRESDEN, Swarthmore College
W. L. DUREN, JR., Tulane University
P. S. DWYER, University of Michigan
L. A. DYE, The Citadel

E. D. EAVES, University of Tennessee
P. D. EDWARDS, Ball State Teachers College
SAMUEL EILENBERG, Indiana University
BENJAMIN EPSTEIN, Carnegie Institute of Technology
W. S. ERICKSON, Bell Aircraft Corporation
H. S. EVERETT, University of Chicago

WILLIAM FELLER, Cornell University
F. A. FICKEN, University of Tennessee
N. J. FINE, Washington, D. C.
C. D. FIRESTONE, Cornell University
K. W. FOLLEY, Wayne University
L. R. FORD, Illinois Institute of Technology
R. M. FOSTER, Polytechnic Institute of Brooklyn
A. H. FOX, Union College
J. S. FRAME, Michigan State College
ORRIN FRINK, JR., Pennsylvania State College

M. G. GABA, University of Nebraska
H. M. GEHMAN, University of Buffalo
ABE GELBART, Syracuse University
B. H. GERE, U. S. Naval Academy
J. J. GERGEN, Duke University
IRVING GERST, Control Instrument Co.
J. H. GIESE, Aberdeen Proving Ground
B. P. GILL, College of the City of New York
J. W. GIVENS, Northwestern University
A. M. GLEASON, Harvard University
MICHAEL GOLDBERG, Bureau of Ordnance, Navy Department

MICHAEL GOLOMB, Purdue University
M. O. GONZÁLEZ, Havana University
A. W. GOODMAN, Columbia University
M. J. GOTTLIEB, Washington University
V. G. GROVE, Michigan State College
N. G. GUNDERSON, University of Rochester

THEODORE HAILPERIN, Lehigh University
P. R. HALMOS, University of Chicago
R. W. HAMMING, Bell Telephone Laboratories
J. R. HANNA, University of Wichita
W. J. HARRINGTON, Pennsylvania State College
BERTHA I. HART, Aberdeen Proving Ground
T. W. HATCHER, Virginia Polytechnic Institute
CHARLES HATFIELD, JR., Georgetown, Ky.
C. E. HEILMAN, Elizabethtown College
M. H. HEINS, Brown University
R. G. HELSEL, Ohio State University
FRITZ HERZOG, Michigan State College
M. R. HESTENES, University of Chicago
E. H. C. HILDEBRANDT, Northwestern University
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College
E. W. HOLT, Muhlenberg College
R. H. HOSKINS, John Hancock Mutual Life Insurance Co.
R. C. HUFFER, Beloit College
H. K. HUGHES, Purdue University
RALPH HULL, University of Nebraska
W. A. HURWITZ, Cornell University
L. C. HUTCHINSON, Polytechnic Institute of Brooklyn

R. L. JEFFERY, Queen's University
E. D. JENKINS, Eastern Kentucky State Teachers College
S. A. JENNINGS, University of British Columbia
EVAN JOHNSON, JR., Pennsylvania State College
R. E. JOHNSON, Mount Holyoke College
ROBERTA F. JOHNSON, Wilson College
W. L. JOHNSON, Mississippi Southern College
L. S. JOHNSTON, University of Detroit
A. W. JONES, Michigan State College
B. W. JONES, Cornell University
HARRIS JONES, U. S. Military Academy
P. S. JONES, University of Michigan

MARK KAC, Cornell University
G. K. KALISCH, Cornell University
IRVING KAPLANSKY, University of Chicago
J. L. KELLEY, University of Chicago

A. J. KEMPNER, University of Colorado
J. R. KLINE, University of Pennsylvania
T. L. KOEHLER, Muhlenberg College
H. L. KRALL, Pennsylvania State College

GILLIE A. LAREW, Randolph-Macon Woman's College
C. G. LATIMER, University of Kentucky
MARY A. LEE, Cornell University
SOLOMON LEFSCHETZ, Princeton University
JOSEPH LEHNER, Hydrocarbon Research
WALTER LEIGHTON, Washington University
W. J. LEVEQUE, Cornell University
HARRY LEVY, University of Illinois
F. A. LEWIS, University of Alabama
C. C. LIN, Brown University
Z. L. LOFLIN, Southwestern Louisiana Institute
L. L. LOWENSTEIN, Alfred University
C. I. LUBIN, University of Cincinnati

C. C. MACDUFFEE, University of Wisconsin
SAUNDERS MAC LANE, Harvard University
M. S. MACPHAIL, Acadia University
MORRIS MARDEN, University of Wisconsin
ELLA MARTH, Harris Teachers College
M. H. MARTIN, University of Maryland
V. O. MCBRIEN, College of the Holy Cross
DOROTHY MCCOY, Belhaven College
N. H. MCCOY, Smith College
S. W. MCCUSKEY, Case School of Applied Science
EDITH A. MCDOUGLE, University of Delaware
E. J. MCSHANE, University of Virginia
A. E. MEDER, JR., Rutgers University
L. E. MEHLENBACHER, University of Detroit
N. S. MENDELSOHN, Queen's University
L. L. MERRILL, Stromberg Carlson Company
A. N. MILGRAM, University of Notre Dame
E. D. MILLER, Yuba Junior College
NORMAN MILLER, Queen's University
KNOX MILLSAPS, A.A.F., Wright Field
B. E. MITCHELL, Millsaps College
JOSEPHINE M. MITCHELL, Connecticut College
E. C. MOLINA, East Orange, N. J.
C. N. MOORE, University of Cincinnati
T. W. MOORE, U. S. Naval Academy
RICHARD MORRIS, Rutgers University
D. S. MORSE, Union College
MARSTON MORSE, Institute for Advanced Study
IRMA R. MOSES, Cornell University
C. W. MUNSHOWER, Colgate University
W. R. MURRAY, Franklin and Marshall College
J. R. MUSSELMAN, Western Reserve University

A. L. NELSON, Wayne University
W. V. NEVINS, III, Alfred University
C. V. NEWSOM, Oberlin College

E. G. OLDS, Carnegie Institute of Technology
L. F. OLLMANN, Hofstra College
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Rooms for members of the organizations and their families were provided in two of the University dormitories, and meals were served in the cafeterias in Willard Straight Hall and Martha Van Rensselaer Hall. Tea was served on Monday and Thursday afternoons in Willard Straight Hall by the ladies of the Cornell Mathematics Department.

On Tuesday afternoon Professor Emeritus W. B. Ford and Mrs. Ford invited the visitors for tea at their beautiful home on the west shore of Cayuga Lake about twenty-five miles north of Ithaca. A very enjoyable concert was given on Tuesday evening in the Memorial Room of Willard Straight Hall by three members of the Walden String Quartette, Messrs. Homer Schmitt, Bernard Goodman, and Eugene Weigel, and a pianist, Miss Margaret Squire, all of the Department of Music of Cornell University. On Wednesday afternoon some of the visitors journeyed to Skaneateles for dinner at The Krebs, while others attended a picnic at the Robert H. Treman State Park (Enfield Glen).

A joint dinner for the three organizations was held on Thursday evening in the Ivy Room of Willard Straight Hall, after which the guests moved to the Memorial Room for a brief program of talks. Professor Allendoerfer acted as toastmaster. An address of welcome to the visitors by Provost Arthur S.

Adams of Cornell University was followed by short and interesting talks by Professors P. A. Smith, L. R. Ford, and Harold Hotelling, representing the Society, the Association, and the Institute respectively. Resolutions were presented by Professor C. C. Camp, and adopted by a rising vote, expressing the thanks and appreciation of the three visiting organizations to President Day and the administration of Cornell University; to the members of the Walden String Quartette and Miss Margaret Squire, pianist; to Professor and Mrs. W. B. Ford; to the ladies of the Mathematics Department of Cornell University; and to Professor Jones and the other members of the Committee on Arrangements.

The American Mathematical Society held its sessions between noon Tuesday and noon Friday. The twenty-eighth Colloquium consisted of four lectures on "Topology of smooth manifolds," by Professor Hassler Whitney of Harvard University. Professor W. H. Stone of Harvard gave his retiring presidential address on "Some general principles of functional representation," and Professor J. L. Doob of the University of Illinois gave an address by invitation on "Probability in function space."

The Institute of Mathematical Statistics held a joint session with the Society on Thursday morning, and a session on Friday morning.

The program committee for the Association consisted of Professors J. J. Gergen, chairman, H. L. Dorwart, and E. H. C. Hildebrandt, and this committee arranged the programs for the two sessions held Monday afternoon and Tuesday morning.

First Session of the Association

1. "The Army Universities in Europe, a preview of the veteran's problem," by Professor H. M. Gehman, University of Buffalo.
2. "Curves," by Lt. Col. R. C. Yates, A.U.S., U. S. Military Academy.
3. "Visual aids to the teaching of calculus," by Professor E. A. Whitman, Carnegie Institute of Technology.

Second Session of the Association

1. "Some identities in the theory of determinants," by Professor G. B. Price, University of Kansas.
2. "An elementary proof of Cauchy's integral theorem," by Professor D. V. Widder, Harvard University.
3. "Analytic extension and summability," by Professor R. P. Agnew, Cornell University.

Meeting of the Board of Governors

The Board met on Monday at 8:00 P.M., in the North Room of Willard Straight Hall, thirteen members of the Board being present. Among the more important items of business transacted were the following:

A list of sixty-two applicants for membership, as given below, was presented, and the applicants were elected by the Board.

J. J. ANDREWS, A.M.(St. Louis Univ.) Lecturer, St. Louis Univ., St. Louis, Mo.

S. D. BERNARDI, M.Ed.(Springfield Coll.), M.S.(Notre Dame) Instr., Lehigh Univ., Bethlehem, Pa.

ERWIN BISER, Ph.D.(Pennsylvania) Instr., Univ. of Kansas City, Kansas City, Mo.

LEE BYRNE, Ph.D.(Columbia) Instr., Rutgers Univ., New Brunswick, N. J.

A. B. CARSON, Ph.D.(Chicago) Prof., Army Air Forces Inst. of Tech., Wright Field, Dayton, Ohio

J. A. CLARKSON, Ph.D.(Brown) Asst. Prof., Univ. of Pennsylvania, Philadelphia, Pa.

F. F. CUMMINGS, M.S.(Cornell) Teacher, High School, Bradford, Pa.

MARJORY R. CURTIS, A.B.(Hope Coll.) Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

REV. A. F. DAVIS, A.B.(Catholic Univ.) St. Paul's Friary, Graymoor, Garrison, N. Y. *Grad. Student, Catholic Univ., Washington, D. C.*

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K. E. GORSLINE, A.M.(Denver) Teacher, East High School, Denver, Colo.

ALICE GRAEBER, A.B.(California) Teacher, Bates School, San Francisco, Calif.

W. H. HAMILTON, A.M.(Columbia) Head of Dept., High School, Highland Park, N. J.

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ELIZABETH L. LAHTI, A.B.(N. Mich. Coll. of Educ.) Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

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MAX LONNER, B.S.(C.C.N.Y.) Admin. Asst.; Part-time teacher, Hebrew Parochial High School, Brooklyn, N. Y.

C. J. MALONEY, A.M.(Minnesota) Research Asso., Iowa State Coll., Ames, Iowa

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TAKASHI TERAMI, Ph.D.(California) Instr., Coll. of St. Thomas, St. Paul, Minn.

BROTHER L. THOMAS, M.S.(St. Louis Univ.) Christian Brothers Coll., St. Louis, Mo. *Grad. Fellow, St. Louis Univ.*

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REV. G. W. WALKER, B.D.(Auburn Theol. Sem.), A.M.(Hamilton Coll.) Pastor, Presbyterian Church, Perry, N. Y.

MARY S. WILSON, B.S.(Georgia) Grad. Student, Univ. of Georgia, Athens, Ga.

R. L. WILSON, A.M.(Wisconsin) Grad. Asst., Univ. of Wisconsin, Madison, Wis.

MRS. REBECCA W. WINCHESTER, A.B.(Greensboro Coll.) Grad. Student, Univ. of Michigan, Ann Arbor, Mich.

E. A. ZUBAY, A.M.(Minnesota) Instr., Drake Univ., Des Moines, Iowa

The Board voted to accept the invitation of Yale University to hold the Summer Meeting of 1947 at New Haven, Connecticut, and the invitation from the University of Georgia to hold the Annual Meeting of 1947 at Athens, Georgia.

Professor Dresden described the work of the Committee on Aid to Devastated Libraries, and the Board voted to donate back numbers of the MONTHLY, of value not to exceed $250, to complete broken sets which might be given to the Committee, and that copies of Carus Monographs might be included in this donation.

A letter was read from Chairman R. G. Harrison of the National Research Council concerning the adherence of the United States to the United Nations Educational, Scientific, and Cultural Organization, UNESCO. The Board authorized our representative on the National Research Council, R. E. Langer, to write a letter expressing the interest of the Association in the organization and our desire to cooperate in establishing good international relations in the field of mathematics.

The Secretary reported that the Editors of the Annals of Mathematics were announcing an increase in their subscription rate, and that they would no longer be able to offer the special rate of $2.50 per year to members of the

Association. The Board voted their thanks to the Annals for their generous treatment of members of the Association in the past. Since the Annals would be offered to members of the American Mathematical Society at a slightly reduced rate, it was felt that no special arrangement with the Association would be necessary.

The Board voted to authorize the Carus Monograph Committee to contract with the George Banta Publishing Company for a thousand copies of the sixth monograph, Fourier Series and Orthogonal Polynomials, by Dunham Jackson, to be made by the photo-offset process.

President MacDuffee and Professor A. A. Bennett as representatives of the Association had attended a recent conference in Washington, sponsored by the American Council on Education, on the relations of higher education to the federal government. They presented to the Board a report of the conference, and expressed the opinion that it had been very effective, and that our membership in the American Council on Education was of definite value to us.

The Board voted an appropriation of an amount not to exceed $50.00 to the Wisconsin Section to cover the expense of clerical help, postage, *etc.*, in the work of a committee and several sub-committees which are undertaking a study of mathematical education in that state.

WALTER B. CARVER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Peoria, May 9–10, 1947
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
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WISCONSIN, Madison, May, 1947

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